EXAM 2

Math 216, 2012-2013 Spring, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.
All answers must be simplified. All of the policies and guidelines
on the class webpages are in effect on this exam.

Good luck!

Name ______________________________

Disc.: Number__________ TA____________________ Day/Time____________________

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ____________________

1. __________

2. __________

3. __________

4. __________

5. __________

6. __________

Total Score __________ (/100 points)

1. *(14 pts)* In this problem, we consider the five functions

$$\begin{aligned} f_1 &= 4\sin(x - \pi/2) \\ f_2 &= 3\sin(x - \pi/8) \\ f_3 &= 6\cos(x) - 3\sin(x - \pi/3) \\ f_4 &= 2\sin(x + \pi/6) \\ f_5 &= \cos(x) - \sin(x) \end{aligned}$$

Without using the Wronskain, decide if the collection $\{f_1, f_2, f_3, f_4, f_5\}$ is linearly independent or linearly dependent, and explain how you know.

2. *(18 pts)* Find a fundamental set of solutions to the differential equation

$$y^{[5]} - 4y^{[4]} + 28y''' + 4y'' - 29y' = 0$$

(Hint: You may use the fact that $e^{2x}\cos(5x)$ *is a solution.)*

3. *(14 pts)* What is the form of a particular solution to the differential equation below? (Do not solve for the coefficients.)

$$y^{[4]} + 2y'' + y = xe^{2x}\sin(3x) + \cos(x) + 5$$

4. *(12 pts)* The image of the linear transformation $T : P_4 \to \mathbb{R}^7$ is a 3-dimensional subspace of $\mathbb{R}^7$ (Recall that P_4 is the vector space of all real polynomials of degree 4 or less). What is the dimension of the kernel of T? (Make sure to explain your reasoning!)

5. *(24 pts)* In this problem you will be guided through a process that will result in finding a fundamental set of solutions for the equation below on the interval $(0, \infty)$.

$$L(y) = y'' + \left(3 - \frac{2}{x}\right) y' - \frac{6y}{x} = 0$$

Note that

$$\begin{aligned} L &= D^2 + \left(3 - \frac{2}{x}\right) D - \frac{6}{x} \\ &= \left(D - \frac{2}{x}\right)(D+3) \end{aligned}$$

(a) What is the dimension of the space of solutions to $L(y) = 0$?

(b) Find the kernel of $(D+3)$ (that is, solve $y' + 3y = 0$), and explain how you know these functions must also solve $L(y) = 0$.

(c) Confirm that x^2 is in the kernel of $\left(D - \frac{2}{x}\right)$.

(d) Your friend Bob argues that, since $\left(D - \frac{2}{x}\right)(x^2) = 0$, it also follows that

$$L(x^2) = \left(D - \frac{2}{x}\right)(D+3)(x^2) = (D+3)\left(D - \frac{2}{x}\right)(x^2) = (D+3)(0) = 0$$

Bob thus claims that x^2 is a solution to $L(y) = 0$. Is his reasoning valid? Explain.

(e) Find any solution to $(D+3)(y) = x^2$.

(f) Show that the solutions to $(D+3)(y) = x^2$ are also solutions to $L(y) = 0$.

(g) Combine the previous results to form a fundamental set of solutions to the original equation.

6. *(18 pts)* The functions y_1, y_2, y_3 are solutions to the linear differential equation

$$L(y) = q_3(x)y''' + q_2(x)y'' + q_1(x)y' + q_0(x)y = 0$$

where all of the q_i functions are known to be continuous, and q_3 is never zero.

It is also known that

$$M = \begin{pmatrix} y_1(0) & y_2(0) & y_3(0) \\ y_1'(0) & y_2'(0) & y_3'(0) \\ y_1''(0) & y_2''(0) & y_3''(0) \end{pmatrix} = \begin{pmatrix} 1 & 1 & -2 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$$

(a) Show that $u(x) = y_1 + y_2 + y_3$ is a solution to the equation $L(y) = 0$.

(b) Find the initial values $u(0)$, $u'(0)$, $u''(0)$. *(Hint: Think about a vector that you might multiply by the matrix M.)*

(c) Use the initial values from part (b) to show that u must be the zero function.

(d) What does the result of the previous part tell you about the independence or dependence of $\{y_1, y_2, y_3\}$?