EXAM 1

Math 216, 2012-2013 Spring, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.
All answers must be simplified. All of the policies and guidelines
on the class webpages are in effect on this exam.

Good luck!

Name Solutions

Disc.: Number________ TA________________ Day/Time________________

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ________________

1. ________

2. ________

3. ________

4. ________

5. ________

6. ________

7. ________

Total Score ________ (/100 points)

1. *(10 pts)* Compute the product indicated below by interpreting the rows of the product as linear combinations (and show the steps in that calculation).

$$\begin{pmatrix} 3 & 4 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 7 & 8 \\ 6 & 5 \end{pmatrix}$$

1st row $= 3\begin{pmatrix} 7 & 8 \end{pmatrix} + 4\begin{pmatrix} 6 & 5 \end{pmatrix}$

$= \begin{pmatrix} 21 & 24 \end{pmatrix} + \begin{pmatrix} 24 & 20 \end{pmatrix} = \begin{pmatrix} 45 & 44 \end{pmatrix}$

2nd row $= 2\begin{pmatrix} 7 & 8 \end{pmatrix} + 1\begin{pmatrix} 6 & 5 \end{pmatrix}$

$= \begin{pmatrix} 14 & 16 \end{pmatrix} + \begin{pmatrix} 6 & 5 \end{pmatrix} = \begin{pmatrix} 20 & 21 \end{pmatrix}$

product $= \boxed{\begin{pmatrix} 45 & 44 \\ 20 & 21 \end{pmatrix}}$

2. *(15 pts)* Compute the determinant of the matrix below (do not use a cofactor expansion!) by using antisymmetry (more than once) and the theorem from class about upper triangular matrices.

$$\begin{pmatrix} 5 & 2 & 0 \\ 6 & 3 & 4 \\ 0 & 7 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 5 & 2 & 0 \\ 6 & 3 & 4 \\ 0 & 7 & 0 \end{pmatrix} \xrightarrow[\text{rows } 1,2]{\text{switch}} \begin{pmatrix} 6 & 3 & 4 \\ 5 & 2 & 0 \\ 0 & 7 & 0 \end{pmatrix} \xrightarrow[\text{cols } 1,3]{\text{switch}} \begin{pmatrix} 4 & 3 & 6 \\ 0 & 2 & 5 \\ 0 & 7 & 0 \end{pmatrix} \xrightarrow[\text{cols } 2,3]{\text{switch}} \begin{pmatrix} 4 & 6 & 3 \\ 0 & 5 & 2 \\ 0 & 0 & 7 \end{pmatrix}$$

So $\det = (-1)^3 \det\begin{pmatrix} 4 & 6 & 3 \\ 0 & 5 & 2 \\ 0 & 0 & 7 \end{pmatrix}$ applying antisymmetry 3 times

$= -(4 \cdot 5 \cdot 7)$ prod. of diag. ents of upper tri matrix

$= \boxed{-140}$

3. *(15 pts)* Suppose you know that the matrix A is nonsingular (that is, that rref(A) is the identity matrix). Use elementary matrices to prove that A must be invertible.

The row reduction of A to $R = I$ (b/c A is nonsingular) can be represented with elementary matrices

as

$$E_k \cdots E_2 E_1 A = I$$

where each E_i represents a row operation.

Using associativity and setting $E = E_k \cdots E_2 E_1$,

$$(E_k \cdots E_2 E_1) A = I$$

$$EA = I$$

Then $E = A^{-1}$, and A is invertible.

4. *(15 pts)* Consider the question (Q): "Is the collection of vectors below linearly independent or linearly dependent?"

$$\left\{ \begin{pmatrix} 5 \\ 2 \\ 7 \end{pmatrix}, \begin{pmatrix} 2 \\ -7 \\ 3 \end{pmatrix}, \begin{pmatrix} -7 \\ -3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix} \right\}$$

(a) What is the system of linear equations whose set of solutions decides the question (Q)?

$$c_1 \begin{pmatrix} 5 \\ 2 \\ 7 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ -7 \\ 3 \end{pmatrix} + c_3 \begin{pmatrix} -7 \\ -3 \\ 1 \end{pmatrix} + c_4 \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Longleftrightarrow \boxed{\begin{aligned} 5c_1 + 2c_2 - 7c_3 + 1c_4 &= 0 \\ 2c_1 - 7c_2 - 3c_3 + 0c_4 &= 0 \\ 7c_1 + 3c_2 + 1c_3 + 4c_4 &= 0 \end{aligned}}$$

(b) What specifically about the set of solutions to that system would resolve the question (Q)?

If there is a non-trivial solution, then $\Rightarrow$ dependent

If there is not, then $\Rightarrow$ independent

(c) What is the answer to the question (Q)? (You may support your answer with any valid reasoning (no appeals to geometric intuition, or citing theorems from the book), even if you do not actually solve the system.)

The coefficient matrix has 3 rows so at most 3 pivots, and thus a column with no pivot. So there is a free variable; and of course $\vec{0}$ is a solution. Thus there are ∞'ly many solutions, and thus a non-trivial solution. So: dependent.

5. *(15 pts)* Your friend Bob has found that the complete set of solutions to the system $A\vec{x} = \vec{b}_1$ is given by $\begin{pmatrix} 3 - 4x_2 + 5x_4 \\ x_2 \\ 2 + x_4 \\ x_4 \end{pmatrix}$, and that $\begin{pmatrix} 2 \\ 6 \\ 8 \\ 3 \end{pmatrix}$ is a particular solution to the system $A\vec{x} = \vec{b}_2$.

(a) Show that $\begin{pmatrix} 3 \\ 0 \\ 2 \\ 0 \end{pmatrix}$ is a particular solution to $A\vec{x} = \vec{b}_1$.

Let $x_2 = 0$, $x_4 = 0$ in the given set.

(b) Find the complete set of homogeneous solutions for the matrix A.

$\{\text{complete set}\} = \vec{x}_p + \{\text{homogeneous}\}$

So $\{\text{homogeneous}\} = \{\text{complete set}\} - \vec{x}_p$

$$= \boxed{\begin{pmatrix} -4x_2 + 5x_4 \\ x_2 \\ x_4 \\ x_4 \end{pmatrix}}$$

(c) Find a particular solution to $A\vec{x} = \vec{b}_1 + 3\vec{b}_2$.

$A\begin{pmatrix} 3 \\ 0 \\ 2 \\ 0 \end{pmatrix} = \vec{b}_1$, $A\begin{pmatrix} 2 \\ 6 \\ 8 \\ 3 \end{pmatrix} = \vec{b}_2$

Then $A\left(\begin{pmatrix} 3 \\ 0 \\ 2 \\ 0 \end{pmatrix} + 3\begin{pmatrix} 2 \\ 6 \\ 8 \\ 3 \end{pmatrix}\right) = \vec{b}_1 + 3\vec{b}_2$

$$\boxed{A\begin{pmatrix} 9 \\ 18 \\ 26 \\ 9 \end{pmatrix} = \vec{b}_1 + 3\vec{b}_2}$$

(d) Find the complete set of solutions to the system $A\vec{x} = \vec{b}_1 + 3\vec{b}_2$.

$\{\text{complete set}\} = \vec{x}_p + \{\text{homogeneous}\}$

$$= \boxed{\begin{pmatrix} 9 \\ 18 \\ 26 \\ 9 \end{pmatrix} + \begin{pmatrix} -4x_2 + 5x_4 \\ x_2 \\ x_4 \\ x_4 \end{pmatrix}}$$

6. *(15 pts)* In this problem you will prove one case of Cramer's Rule. Consider the system below, where it is known that the coefficient matrix is nonsingular. *(Hint: Recall that $\vec{b}$ is a linear combination of the columns of the coefficient matrix, and use multilinearity.)*

$$A = \begin{pmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{pmatrix} \longrightarrow \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \longleftarrow \vec{b}$$

Show that

$$x_2 = \frac{\det \begin{pmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{pmatrix}}{\det A}$$

$$\vec{b} = x_1 \vec{a}_1 + x_2 \vec{a}_2 + x_3 \vec{a}_3$$

Then by multilinearity in the second column:

$$\det \begin{pmatrix} \vec{a}_1 & \vec{b} & \vec{a}_3 \end{pmatrix} = x_1 \det \begin{pmatrix} \vec{a}_1 & \vec{a}_1 & \vec{a}_3 \end{pmatrix} + x_2 \det \begin{pmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{pmatrix} + x_3 \det \begin{pmatrix} \vec{a}_1 & \vec{a}_3 & \vec{a}_3 \end{pmatrix}$$

$$= 0 + x_2 \det A + 0$$

So
$$x_2 = \frac{\det \begin{pmatrix} \vec{a}_1 & \vec{b} & \vec{a}_3 \end{pmatrix}}{\det A} = \frac{\det \begin{pmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{pmatrix}}{\det A}$$

7. *(15 pts)* One of the requirements for a set V (with operations "$\oplus$" and "$\odot$") to be a vector space is that, for all real scalars c and all vectors $\vec{v}$ and $\vec{w}$,

$$c \odot (\vec{v} \oplus \vec{w}) = (c \odot \vec{v}) \oplus (c \odot \vec{w})$$

Consider the set V of all 2×2 matrices, with operations defined by

$$\begin{aligned} A \oplus B &= AB \\ c \odot A &= cA \end{aligned}$$

Prove or disprove: The set V with these operations satisfies the previously listed requirement.

<u>False</u> : Counterexample : let $A = B = I$, $c = 2$.

Then $c \odot (A \oplus B) = 2(II) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

but $(c \odot A) \oplus (c \odot B) = (2I)(2I) = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$