EXAM 3

Math 216, 2012-2013 Fall. Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.
All answers must be simplified. All of the policies and guidelines
on the class webpages are in effect on this exam.

Good luck!

Name Solutions

Disc.: Number________ TA________ Day/Time________

1. ________
2. ________
3. ________
4. ________
5. ________
6. ________
7. ________

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ________

Total Score ________ (/100 points)

1. *(10 pts)* Find an eigenvector with eigenvalue 2 for the matrix below.

$$\begin{pmatrix} 5 & 0 & -1 \\ 1 & 3 & -1 \\ 1 & -2 & 3 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 3 & 0 & -1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -2 & 1 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 3 & -2 & 0 \\ 0 & 6 & -4 & 0 \end{array}\right) \begin{array}{l} ③ \\ ② - ③ \\ ① - 3③ \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -2/3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} ① \\ ②/3 \\ ③ - 2② \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -1/3 & 0 \\ 0 & 1 & -2/3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} ① + 2② \\ ② \\ ③ \end{array}$$

$\to x = \frac{1}{3}z, \; y = \frac{2}{3}z$

So an eigenvector is

$$\boxed{\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}}$$

2. *(15 pts)* Show that if a matrix A is real and symmetric, and if $\vec{v}$ is an nonzero eigenvector with eigenvalue λ, then λ must be real. *(Hint: Consider $\langle A\vec{v}, \vec{v}\rangle_H$.)*

$$\langle A\vec{v}, \vec{v}\rangle_H = \langle \vec{v}, A\vec{v}\rangle \quad \text{(b/c } A \text{ is Hermitian)}$$

$$\langle \lambda\vec{v}, \vec{v}\rangle_H = \langle \vec{v}, \lambda\vec{v}\rangle_H$$

$$\lambda\langle \vec{v}, \vec{v}\rangle_H = \bar{\lambda}\langle \vec{v}, \vec{v}\rangle_H$$

(these are nonzero b/c $\vec{v}$ is nonzero)

$$\lambda = \bar{\lambda}$$

So λ must be real.

3. *(15 pts)* Use the idea of diagonalization to find a matrix B for which the vectors below are eigenvectors with eigenvalues 3 and 2 (respectively).

$$\vec{v}_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \quad \text{and} \quad \vec{v}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$[T]_{\mathcal{S}}^{\mathcal{S}} = [I]_{\mathcal{V}}^{\mathcal{S}} \, [T]_{\mathcal{V}}^{\mathcal{V}} \, [I]_{\mathcal{S}}^{\mathcal{V}}$$

$$B = P \quad D \quad P^{-1}$$

$$= \underbrace{\begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}} \underbrace{\begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}^{-1}}$$

$$= \begin{pmatrix} 9 & 4 \\ 12 & 6 \end{pmatrix} \begin{pmatrix} 3 & -2 \\ -4 & 3 \end{pmatrix}$$

$$\boxed{B = \begin{pmatrix} 11 & -6 \\ 12 & -6 \end{pmatrix}}$$

4. *(15 pts)* Find the Jordan canonical form and a Jordan basis for the matrix A below. You may use the fact that the characteristic polynomial is $p(\lambda) = (\lambda - 2)^3$, and that the two augmented matrices listed below are row equivalent as indicated.

$$A = \begin{pmatrix} 9 & 2 & -7 \\ 16 & 6 & -16 \\ 11 & 3 & -9 \end{pmatrix}$$

$$\left(\underbrace{\begin{matrix} 7 & 2 & -7 \\ 16 & 4 & -16 \\ 11 & 3 & -11 \end{matrix}}_{A-\lambda I} \left| \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \right.\right) \text{ is row equivalent to } \left(\underbrace{\begin{matrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{matrix}}_{R=\text{rref}(A-\lambda I)} \left| \underbrace{\begin{matrix} 0 & 3/4 & -1 \\ 0 & -11/4 & 4 \\ 1 & 1/4 & -1 \end{matrix}}_{E} \right.\right)$$

Find eigenvectors: $(A-\lambda I|\vec{0})$ reduces to $(R|E\vec{0})$
$= (R|\vec{0})$

Yields eigenvector $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \vec{v}_1$

1 eigenvector $\Rightarrow$ 1 basic Jordan block $\Rightarrow$

$$\boxed{J = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}}$$

To find $\vec{v}_2, \vec{v}_3$, solve $(A-\lambda I)\vec{v}_i = \vec{v}_{i-1}$

or $R\vec{v}_i = E\vec{v}_{i-1}$:

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\vec{v}_2 = E\vec{v}_1 = \begin{pmatrix} -1 \\ 4 \\ 0 \end{pmatrix} \quad \text{Choose } \vec{v}_2 = \begin{pmatrix} -1 \\ 4 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\vec{v}_3 = E\vec{v}_2 = \begin{pmatrix} 3 \\ -11 \\ 0 \end{pmatrix} \quad \text{Choose } \vec{v}_3 = \begin{pmatrix} 3 \\ -11 \\ 0 \end{pmatrix}$$

So $\boxed{\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ -11 \\ 0 \end{pmatrix} \right\}}$ is a Jordan basis.

5. *(15 pts)* Complete the Gram-Schmidt orthonormalization of the basis $\mathcal{V} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ of a subspace of $\mathbb{R}^4$ (using the usual dot product), noting that the first two vectors in this list are already orthonormal.

$$\vec{v}_1 = \begin{bmatrix} 2/7 \\ 3/7 \\ 0 \\ 6/7 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} -6/7 \\ 0 \\ 3/7 \\ 2/7 \end{bmatrix} \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

As noted, can choose $\boxed{\vec{u}_1 = \vec{v}_1, \quad \vec{u}_2 = \vec{v}_2}$

$$\vec{x}_3 = \vec{v}_3 - (\vec{u}_1 \cdot \vec{v}_3)\vec{u}_1 - (\vec{u}_2 \cdot \vec{v}_3)\vec{u}_2$$

$$= \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} - \left(\begin{pmatrix} 2/7 \\ 3/7 \\ 0 \\ 6/7 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}\right)\begin{pmatrix} 2/7 \\ 3/7 \\ 0 \\ 6/7 \end{pmatrix} - \left(\begin{pmatrix} -6/7 \\ 0 \\ 3/7 \\ 2/7 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}\right)\begin{pmatrix} -6/7 \\ 0 \\ 3/7 \\ 2/7 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} - \left(\frac{32}{7}\right)\begin{pmatrix} 2/7 \\ 3/7 \\ 0 \\ 6/7 \end{pmatrix} - \left(\frac{11}{7}\right)\begin{pmatrix} -6/7 \\ 0 \\ 3/7 \\ 2/7 \end{pmatrix}$$

$$= \left(\begin{pmatrix} 49 \\ 98 \\ 147 \\ 196 \end{pmatrix} - \begin{pmatrix} 64 \\ 96 \\ 0 \\ 192 \end{pmatrix} - \begin{pmatrix} -66 \\ 0 \\ 33 \\ 22 \end{pmatrix}\right) \Big/ 49$$

$$= \begin{pmatrix} 51 \\ 2 \\ 114 \\ -18 \end{pmatrix} \Big/ 49$$

$$\vec{u}_3 = \frac{\vec{x}_3}{\|\vec{x}_3\|} = \boxed{\begin{pmatrix} 51 \\ 2 \\ 114 \\ -18 \end{pmatrix} \Big/ \sqrt{51^2 + 2^2 + 114^2 + 18^2}}$$

6. *(15 pts)* Suppose that for a matrix B, we have three eigenvectors (below) with eigenvalues 3, -2, and 4 (respectively).

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \vec{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

(a) Find a fundamental set of solutions for the equation $\vec{y}' = B\vec{y}$.

$$e^{xB}\vec{V}_1 = e^{3x}\vec{V}_1 = e^{3x}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} e^{3x} \\ e^{3x} \\ 0 \end{pmatrix}$$

$$e^{xB}\vec{V}_2 = e^{-2x}\vec{V}_2 = e^{-2x}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} e^{-2x} \\ 0 \\ e^{-2x} \end{pmatrix}$$

$$e^{xB}\vec{V}_3 = e^{4x}\vec{V}_3 = e^{4x}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ e^{4x} \\ e^{4x} \end{pmatrix}$$

(b) Compute the matrix e^{xB} *(Hint: Recall that the i^{th} column is $e^{xB}\vec{e}_i$, and note that each $\vec{e}_i$ is an easy linear combination of the eigenvectors.)*

$$\text{1st col} = e^{xB}\vec{e}_1 = e^{xB}\left(\frac{\vec{V}_1+\vec{V}_2-\vec{V}_3}{2}\right) = \begin{pmatrix} e^{3x}+e^{-2x} \\ e^{3x}-e^{4x} \\ e^{-2x}-e^{4x} \end{pmatrix}/2$$

$$\text{2nd col} = e^{xB}\vec{e}_2 = e^{xB}\left(\frac{\vec{V}_1-\vec{V}_2+\vec{V}_3}{2}\right) = \begin{pmatrix} e^{3x}-e^{-2x} \\ e^{3x}+e^{4x} \\ -e^{-2x}+e^{4x} \end{pmatrix}/2$$

$$\text{3rd col} = e^{xB}\vec{e}_3 = e^{xB}\left(\frac{-\vec{V}_1+\vec{V}_2+\vec{V}_3}{2}\right) = \begin{pmatrix} -e^{3x}+e^{-2x} \\ -e^{3x}+e^{4x} \\ e^{-2x}+e^{4x} \end{pmatrix}/2$$

$$e^{xB} = \begin{pmatrix} e^{3x}+e^{-2x} & e^{3x}-e^{-2x} & -e^{3x}+e^{-2x} \\ e^{3x}-e^{4x} & e^{3x}+e^{4x} & -e^{3x}+e^{4x} \\ e^{-2x}-e^{4x} & -e^{-2x}+e^{4x} & e^{-2x}+e^{4x} \end{pmatrix}/2$$

(c) Solve the initial value problem $\vec{y}' = B\vec{y}$, with $\vec{y}(0) = (2, 5, 7)$.

$$\vec{y} = e^{xB}\vec{y}_0 = e^{xB}\begin{pmatrix} 2 \\ 5 \\ 7 \end{pmatrix} = \begin{pmatrix} 2e^{-2x} \\ 5e^{4x} \\ 2e^{-2x}+5e^{4x} \end{pmatrix}$$

7. *(15 pts)* Use the arithmetic indicated below to find a fundamental set of solutions to the system $\vec{y}' = A\vec{y}$. *(Hint: You may find it useful to note that two of the matrices expressed below are both transposes and inverses of each other.)*

$$A = \begin{pmatrix} 86/49 & 13/49 & 18/49 \\ 6/49 & 116/49 & 40/49 \\ -45/49 & 12/49 & 92/49 \end{pmatrix} = \underbrace{\begin{pmatrix} 2/7 & 3/7 & -6/7 \\ 6/7 & 2/7 & 3/7 \\ -3/7 & 6/7 & 2/7 \end{pmatrix}}_{P} \underbrace{\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}}_{J} \underbrace{\begin{pmatrix} 2/7 & 6/7 & -3/7 \\ 3/7 & 2/7 & 6/7 \\ -6/7 & 3/7 & 2/7 \end{pmatrix}}_{P^{-1}}$$

As noted, a Jordan basis is $\left\{ \underbrace{\begin{pmatrix} 2/7 \\ 6/7 \\ -3/7 \end{pmatrix}}_{\vec{v}_1}, \underbrace{\begin{pmatrix} 3/7 \\ 2/7 \\ 6/7 \end{pmatrix}}_{\vec{v}_2}, \underbrace{\begin{pmatrix} -6/7 \\ 3/7 \\ 2/7 \end{pmatrix}}_{\vec{v}_3} \right\}$

eigenvector!

$$e^{xA}\vec{v}_1 = e^{2x}\vec{v}_1 = \frac{e^{2x}}{7}\begin{pmatrix} 2 \\ 6 \\ -3 \end{pmatrix}$$

$$e^{xA}\vec{v}_2 = e^{2x}\left(\vec{v}_2 + x\vec{v}_1\right) = \frac{e^{2x}}{7}\begin{pmatrix} 3+2x \\ 2+6x \\ 6-3x \end{pmatrix}$$

$$e^{xA}\vec{v}_3 = e^{2x}\left(\vec{v}_3 + x\vec{v}_2 + \tfrac{1}{2}x^2\vec{v}_1\right) = \frac{e^{2x}}{7}\begin{pmatrix} -6+3x+x^2 \\ 3+2x+3x^2 \\ 2+6x-\frac{3}{2}x^2 \end{pmatrix}$$

This is a fundamental set of solutions.