EXAM 2

Math 216, 2012-2013 Fall, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.
All answers must be simplified. All of the policies and guidelines
on the class webpages are in effect on this exam.

Good luck!

Name Solutions

Disc.: Number________ TA______________ Day/Time______________

1. ________
2. ________
3. ________
4. ________
5. ________
6. ________
7. ________
8. ________
9. ________

"I have adhered to the Duke Community
Standard in completing this
examination."

Signature: ______________

Total Score ________ (/100 points)

1. *(6 pts)* Can the Wronskian alone (without using other knowledge of these functions) allow us to conclude whether the collection $\{x^2, 2x+5, 3x-7, 2x^2+1\}$ is independent or dependent?

$$W(x) = \det \begin{pmatrix} x^2 & 2x+5 & 3x-7 & 2x^2+1 \\ 2x & 2 & 3 & 4x \\ 2 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix} = 0$$

The Wronskian fails to tell us if these functions are l.i. or l.d.

(Sidenote: if we use the additional knowledge that these functions are analytic, we can conclude from the above that these are l.d.)

2. *(6 pts)* By trial and error, Bob has found three valid solutions to the equation $2y''' - 3y'' + 2y' + y = 0$, and he has correctly computed that the Wronskian is the zero function. Using this information, can you determine if his three functions form an independent set or a dependent set? Explain.

These are 3 solutions, and the LDE given is 3rd order.

The LDE also satisfies the conditions of the existence/uniqueness thm.

By the theorem from class then,

$W(x) = 0 \implies$ these fns are l.d.

3. *(6pts)* Find a real fundamental set of solutions to the differential equation

$$y''' + 2y'' + 4y' + 8y = 0$$

$p(\lambda) = \lambda^3 + 2\lambda^2 + 4\lambda + 8$ possible rat'l roots : $\pm 1, \pm 2, \pm 4, \pm 8$

$p(1) = 15$ ✗ $p(-1) = 5$ ✗ $p(2) = 32$ ✗ $p(-2) = 0$ root!

$$\begin{array}{r|l} & \lambda^2 \qquad\quad +4 \\ \hline \lambda+2 & \lambda^3 + 2\lambda^2 + 4\lambda + 8 \\ & \lambda^3 + 2\lambda^2 \\ \hline & \quad 0 \quad + 4\lambda + 8 \\ & \qquad\quad 4\lambda + 8 \\ \hline & \qquad\qquad 0 \end{array}$$

So $p(\lambda) = (\lambda+2)(\lambda^2+4)$

$= (\lambda+2)(\lambda+2i)(\lambda-2i)$

roots are $-2, 2i, -2i$

This gives as a real fundamental set of solutions

$$\{e^{-2x}, \cos 2x, \sin 2x\}$$

4. (9 pts) Write down the form of the particular solution to the differential equation below (but do NOT solve for the constants).

$$y''' + 2y'' + 4y' + 8y = x^5 e^{-2x}$$

$p(\lambda) = \lambda^3 + 2\lambda^2 + 4\lambda + 8 = (\lambda+2)(\lambda+2i)(\lambda-2i)$ (from prev. prob.)

$r=-2$ is a root, of multiplicity 1. So the form of the part. sol. is

$$\boxed{y_p = x^1\left(c_5x^5 + c_4x^4 + c_3x^3 + c_2x^2 + c_1x + c_0\right)e^{-2x}}$$

5. (5pts) Find the resonant particular solution to the differential equation below.

$$y'' + \omega_0^2 y = \sin\omega_0 t$$

$r = \omega_0 i$ is a root $(m=1)$ of $p(\lambda) = \lambda^2 + \omega_0^2 = (\lambda + \omega_0 i)(\lambda - \omega_0 i)$

Guess: $y_p = At\sin\omega_0 t + Bt\cos\omega_0 t$

$y_p' = A\sin\omega_0 t + At\omega_0\cos\omega_0 t + B\cos\omega_0 t - Bt\omega_0\sin\omega_0 t$

$y_p'' = A\omega_0\cos\omega_0 t + A\omega_0\cos\omega_0 t - At\omega_0^2\sin\omega_0 t$
$\quad - B\omega_0\sin\omega_0 t - B\omega_0\sin\omega_0 t - Bt\omega_0^2\cos\omega_0 t$

Equation becomes:

$$2A\omega_0\cos\omega_0 t - 2B\omega_0\sin\omega_0 t = \sin\omega_0 t$$

so $A=0$, $B = \frac{-1}{2\omega_0}$

and thus $\boxed{y_p = \frac{-t}{2\omega_0}\cos\omega_0 t}$

6. *(11pts)* The function $\delta_a^{[n]} : C^\infty \to \mathbb{R}$ is defined by $\delta_c^{[n]}(f) = f^{[n]}(c)$. Show that this is a linear transformation.

$$\delta_c^{[n]}(af+bg) = (af+bg)^{[n]}(c) = (af^{[n]}+bg^{[n]})(c)$$

$$= a f^{[n]}(c) + b g^{[n]}(c)$$

$$= a\,\delta_c^{[n]}(f) + b\,\delta_c^{[n]}(g) \quad \checkmark$$

7. *(11pts)* Suppose that S and T are linear transformations from a vector space V to a vector space W. Use the definitions of the operations on linear transformations to show that $c(S+T) = cS + cT$.

$$\big(c(S+T)\big)(v) = c\big((S+T)(v)\big) = c\big(S(v)+T(v)\big) = cS(v) + cT(v)$$

$$(cS+cT)(v) = (cS)(v) + (cT)(v) = c(S(v)) + c(T(v)) = cS(v) + cT(v)$$

So $(c(S+T))$ and $(cS+cT)$ always have the same values, so they are equal as linear transformations.

8. *(11pts)* Let $D : C^\infty \to C^\infty$ be defined by $D(f) = f'$. Without citing a previous result, compute $(D-3)^4(x^7e^{3x})$.

$$(D-3)(x^n e^{3x}) = nx^{n-1}e^{3x} + x^n \cdot 3e^{3x} - 3x^n e^{3x} = nx^{n-1}e^{3x}$$

So $(D-3)^4(x^7e^{3x}) = (D-3)^3(7x^6e^{3x}) = (D-3)^2(42x^5e^{3x})$

$$= (D-3)(210x^4e^{3x}) = \boxed{840x^3e^{3x}}$$

9. *(6pts)* Let $f_1 = \sin 2x$, $f_2 = \cos 2x$, $f_3 = 3\cos(2x - \frac{\pi}{3})$, and let V be the vector space with basis $\beta = \{f_1, f_2\}$. The linear transformation $L : V \to V$ is defined by $L(y) = y'' - 2y' + 3y$. Compute the items listed below. *(Hint: Recall that* $\cos(a+b) = \cos a \cos b - \sin a \sin b$.*)*

(a) $[f_3]_\beta$

$$f_3 = 3\cos\left(2x - \pi/3\right) = 3\cos 2x \cos\left(\tfrac{-\pi}{3}\right) - 3\sin 2x \sin\left(-\pi/3\right)$$

$$= \frac{3}{2}\cos 2x + \frac{3\sqrt{3}}{2}\sin 2x$$

So $[f_3]_\beta = \begin{pmatrix} 3\sqrt{3}/2 \\ 3/2 \end{pmatrix}$

(b) $[L]_\beta^\beta$

1st col. $= [L(f_1)]_\beta = [-4\sin 2x - 4\cos 2x + 3\sin 2x]_\beta$

$= [-1\sin 2x - 4\cos 2x]_\beta = \begin{pmatrix} -1 \\ -4 \end{pmatrix}$

2nd col $= [L(f_2)]_\beta = [-4\cos 2x + 4\sin 2x + 3\cos 2x]_\beta$

$= [4\sin 2x - 1\cos 2x]_\beta = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$

So $[L]_\beta^\beta = \begin{pmatrix} -1 & 4 \\ -4 & -1 \end{pmatrix}$

(c) $[L(f_3)]_\beta$ (without computing $L(f_3)$ directly)

$$[L(f_3)]_\beta = [L]_\beta^\beta [f_3]_\beta = \begin{pmatrix} -1 & 4 \\ -4 & -1 \end{pmatrix}\begin{pmatrix} 3\sqrt{3}/2 \\ 3/2 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{3\sqrt{3}}{2} + 6 \\ -6\sqrt{3} - \frac{3}{2} \end{pmatrix}$$