EXAM 1

Math 216, 2012-2013 Fall, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.
All answers must be simplified. All of the policies and guidelines
on the class webpages are in effect on this exam.

Good luck!

Name Solutions

Disc.: Number__________ TA____________________ Day/Time____________________

1. __________

2. __________

3. __________

4. __________

5. __________

6. __________

7. __________

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ____________________

Total Score __________ (/100 points)

1. *(16 pts)* The reduced row echelon form of the augmented matrix for the system $A\vec{x} = \vec{b}$ (where $\vec{x} = (x_1, x_2, x_3, x_4)$), the product E of the elementary matrices used in that row reduction, and E^{-1}, are given below.

$$\text{rref}(A|\vec{b}) = \left(\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 1 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \quad \text{and} \quad E = \begin{pmatrix} 2 & 3 & 2 \\ 4 & -1 & 0 \\ 2 & 1 & 1 \end{pmatrix} \quad \text{and} \quad E^{-1} = \begin{pmatrix} 0.5 & 0.5 & -1 \\ 2 & 1 & -4 \\ -3 & -2 & 7 \end{pmatrix}$$

(a) Find the complete set of solutions to the system $A\vec{x} = \vec{b}$.

$$X_1 + 3X_3 = 2 \Rightarrow X_1 = 2 - 3X_3$$
$$X_2 + 1X_3 = 5 \Rightarrow X_2 = 5 - 1X_3$$

So
$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix} = \begin{pmatrix} 2 - 3X_3 \\ 5 - 1X_3 \\ X_3 \\ X_4 \end{pmatrix} = \boxed{\begin{pmatrix} 2 \\ 5 \\ 0 \\ 0 \end{pmatrix} + X_3 \begin{pmatrix} -3 \\ -1 \\ 1 \\ 0 \end{pmatrix} + X_4 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}}$$

(b) Find a vector $\vec{c}$ for which $A\vec{x} = \vec{c}$ has no solutions.

$A\vec{x} = \vec{c}$ reduces to $EA\vec{x} = E\vec{c}$ or $R\vec{x} = E\vec{c}$, which, due to the row of zeroes in $R = EA$, has no solution when $E\vec{c} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. So we choose

$$\vec{c} = E^{-1}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \boxed{\begin{pmatrix} -1 \\ -4 \\ 7 \end{pmatrix}}$$

(c) Find the original matrix A.

$$R = EA, \text{ so } A = E^{-1}R = \begin{pmatrix} 1/2 & 1/2 & -1 \\ 2 & 1 & -4 \\ -3 & -2 & 7 \end{pmatrix}\begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$A = \boxed{\begin{pmatrix} 1/2 & 1/2 & 2 & 0 \\ 2 & 1 & 7 & 0 \\ -3 & -2 & -11 & 0 \end{pmatrix}}$$

(d) Find bases for the row space, column space, and null space of A.

$$\left\{\begin{pmatrix} -3 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right\} \qquad \left\{\begin{pmatrix} 1 \\ 0 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}\right\} \qquad \left\{\begin{pmatrix} 1/2 \\ 2 \\ -3 \end{pmatrix}, \begin{pmatrix} 1/2 \\ 1 \\ -2 \end{pmatrix}\right\}$$

↑ basis for NS (from (a))

↑ basis for RS (pivot rows of R)

↑ basis for CS (pivot cols of A)

2. *(14 pts)* Use a SINGLE row reduction (you can row reduce anything you choose) to find the reduced row echelon form, determinant, and inverse (or show it does not exist) of the matrix A below. (You must make clear use of the single row reduction in all three computations.)

$$A = \begin{pmatrix} 2 & 4 \\ 1 & 7 \end{pmatrix}$$

$$\left(\begin{array}{cc|cc} 2 & 4 & 1 & 0 \\ 1 & 7 & 0 & 1 \end{array}\right)$$

$\det = -\det(A)$

$$\left(\begin{array}{cc|cc} 1 & 7 & 0 & 1 \\ 2 & 4 & 1 & 0 \end{array}\right) \begin{array}{l} (2) \\ (1) \end{array}$$

$$\left(\begin{array}{cc|cc} 1 & 7 & 0 & 1 \\ 0 & -10 & 1 & -2 \end{array}\right) \begin{array}{l} (1) \\ (2) - 2(1) \end{array}$$

$\det = -\frac{1}{10}\left(-\det(A)\right)$

$$\left(\begin{array}{cc|cc} 1 & 7 & 0 & 1 \\ 0 & 1 & -1/10 & 1/5 \end{array}\right) \begin{array}{l} (1) \\ (2)/-10 \end{array}$$

$$\left(\begin{array}{cc|cc} 1 & 0 & 7/10 & -2/5 \\ 0 & 1 & -1/10 & 1/5 \end{array}\right) \begin{array}{l} (1) - 7(2) \\ (2) \end{array}$$

$\det I = -\frac{1}{10}\left(-\det A\right)$

$\boxed{\mathrm{rref}(A) = I}$, so $\boxed{\text{this is } A^{-1}}$

$\det I = 1$

So $\boxed{\det A = 10}$

3. *(14 pts)* Consider the related matrices below.

$$M = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix} \quad \text{and}$$

$$M_1 = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ 1 & 0 & 0 \\ m_{31} & m_{32} & m_{33} \end{pmatrix} \quad M_2 = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ 0 & 1 & 0 \\ m_{31} & m_{32} & m_{33} \end{pmatrix} \quad M_3 = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ 0 & 0 & 1 \\ m_{31} & m_{32} & m_{33} \end{pmatrix}$$

Suppose also that $\det(M_1) = 3$, $\det(M_2) = 18$, $\det(M_3) = -5$.

(a) Find the simplest possible formula for the determinant of the above matrix M in terms of its entries $\{m_{ij}\}$.

By multilinearity in the 2nd row, we have

$$\det M = m_{21} \det M_1 + m_{22} \det M_2 + m_{23} \det M_3$$

$$\boxed{\det M = 3m_{21} + 18m_{22} - 5m_{23}}$$

(b) Find a similar formula for the determinant of the matrix N given by

$$N = \begin{pmatrix} m_{31} & m_{32} & m_{33} \\ m_{21} & m_{22} & m_{23} \\ m_{11} & m_{12} & m_{13} \end{pmatrix}$$

N is obtained from M by switching two rows (1st & 3rd), so $\det N = -\det M$ by antisymmetry.

$$\boxed{\det N = -3m_{21} - 18m_{22} + 5m_{23}}$$

4. *(14 pts)* The matrix A below can be written as the product $A = M_1 M_2$ of two elementary matrices, M_1 and M_2. Find these elementary matrices M_1 and M_2.

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 7 \end{pmatrix}$$

We row reduce A by two row operations:

$$\begin{pmatrix} 1 & 2 \\ 3 & 7 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{matrix} 1 \\ 2 - 3\,1 \end{matrix} \quad \leftarrow E_1 = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{matrix} 1 - 2\,2 \\ 2 \end{matrix} \quad \leftarrow E_2 = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$$

So the row reduction in matrix form is

$$E_2 E_1 A = I$$

and thus

$$A = E_1^{-1} E_2^{-1}$$

So we can choose $M_1 = E_1^{-1}$, $M_2 = E_2^{-1}$, so

$$\boxed{M_1 = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \qquad M_2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}}$$

5. *(14 pts)* Consider the vectors below in the vector space P_2 of polynomials of degree at most 2. (Do not use here previous knowledge you might have about the dimension of P_2.)

$$\begin{aligned} p_1(x) &= x^2 - 2x \\ p_2(x) &= x^2 + 2x \\ p_3(x) &= x + 1 \\ p_4(x) &= 8x - 5 \end{aligned}$$

Show that $\beta = \{p_1, p_2, p_3\}$ is a basis for P_2, and compute $[p_4]_\beta$.

<u>Note:</u> $c_1 p_1 + c_2 p_2 + c_3 p_3 = b_2 x^2 + b_1 x + b_0$ $(= b)$

is equivalent to $\underbrace{\begin{pmatrix} 1 & 1 & 0 \\ -2 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}}_{A} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} b_2 \\ b_1 \\ b_0 \end{pmatrix}$

<u>Indep</u>: Need to show $c_1 p_1 + c_2 p_2 + c_3 p_3 = 0$ has unique sol.; by above note, need to show rref(A) has pivot in every col. $\det(A) = 4 \neq 0$, so A is nonsingular, giving us the result.

<u>Span</u>: Need to show $c_1 p_1 + c_2 p_2 + c_3 p_3 = b$ has solutions for all $b \in P_2$; by above note, need to show rref (A) has pivot in every row. Again, this is a consequence of nonsingularity of A.

To solve for $[p_4]_\beta$, we solve $c_1 p_1 + c_2 p_2 + c_3 p_3 = p_4$, which by the note is equivalent to

$$\begin{pmatrix} 1 & 1 & 0 \\ -2 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 8 \\ -5 \end{pmatrix}$$

This gives us $\boxed{[p_4]_\beta = \begin{pmatrix} -13/4 \\ 13/4 \\ -5 \end{pmatrix}}$

6. *(14 pts)* Your friend Bob wishes to form a vector space consisting of all invertible 2×2 matrices, using the operations below.

$$\begin{aligned} A \oplus B &= AB \\ c \odot A &= cA \end{aligned}$$

Does this form a vector space? If so, prove it; if not, identify one requirement that fails, and show this failure with an explicit example.

This does not form a vector space. For example, we do not have $A \oplus B = B \oplus A$, because $AB \neq BA$, as illustrated by

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

(Other failures: $c \odot (A \oplus B) \neq c \odot A \oplus c \odot B$

$(c+d) \odot A \neq c \odot A \oplus d \odot A$)

7. *(14 pts)* Let W be the subset of C^∞ consisting of all function of the form Ae^{x+B}, where A and B can be any real numbers.

Your friend Bob says that W can be viewed as a subspace of C^∞; is he right? If he is, what is the dimension of W?

We check that W is closed under addition and scalar mult.:

$$A_1e^{x+B_1} + A_2e^{x+B_2} = A_1e^{B_1}e^x + A_2e^{B_2}e^x$$
$$= \left(A_1e^{B_1} + A_2e^{B_2}\right)e^{x+0}$$

$$c\left(A_1e^{x+B_1}\right) = (cA_1)e^{x+B_1}$$

These are of the given form, and are thus in W. So W is closed under both operations and thus $\boxed{\text{is a subspace}}$.

The algebra above motivates the observation that

$$W = \left\{Ae^{x+B}\right\} = \left\{Ce^x\right\}$$

So $\{e^x\}$ is a basis for W, and thus

$$\boxed{\dim W = 1}$$