EXAM 3

Math 107, 2011-2012 Spring, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.
All answers must be simplified. All of the policies and guidelines
on the class webpages are in effect on this exam.

Good luck!

Name Solutions

Rec: Number__________ TA____________________ Day/Time____________________

"I have adhered to the Duke Community
Standard in completing this
examination."

Signature: ____________________

1. __________

2. __________

3. __________

4. __________

5. __________

Total Score __________ (/100 points)

1. *(20 pts)* Note the following arithmetic:

$$\begin{pmatrix} 2 & 0 & 6 \\ 4 & -3 & -2 \\ -4 & 3 & 0 \end{pmatrix} = \underbrace{\begin{pmatrix} 2 & 12 & 12 \\ -2 & -8 & -11 \\ 2 & 6 & 9 \end{pmatrix}}_{A} \begin{array}{ccc} \underbrace{\vec{v}_1}_{} & \underbrace{\vec{v}_2}_{} & \underbrace{\vec{v}_3}_{} \\ \end{array}\! \begin{pmatrix} 1 & 0 & -3 \\ 2 & -1 & 1 \\ -2 & 1 & 0 \end{pmatrix}$$

(a) Find ALL of the eigenvectors and eigenvalues of the matrix A.

The above arithmetic tells us that the columns $\vec{v}_1, \vec{v}_2, \vec{v}_3$ of are eigenvectors, with

$A\vec{v}_1 = 2\vec{v}_1$

$A\vec{v}_2 = 3\vec{v}_2$

$A\vec{v}_3 = -2\vec{v}_3$

So the eigenvectors and eigenvalues are:

$$\boxed{\begin{array}{l} \vec{v}_1 : \lambda_1 = 2 \\ \vec{v}_2 : \lambda_2 = 3 \\ \vec{v}_3 : \lambda_3 = -2 \end{array}}$$

(b) Find the characteristic polynomial of A, WITHOUT computing a determinant.

Each of the eigenvalues above is a root of p, and thus corresponds to a factor of p. So

$$\boxed{p(\lambda) = (\lambda - 2)(\lambda - 3)(\lambda - (-2))}$$

(c) Viewing A as representing a linear transformation T with respect to the standard basis $\mathcal{S}$, find another basis $\mathcal{V}$ and a diagonal matrix D that represents the same linear transformation T with respect to [redacted] $\mathcal{V}$.

Given these three equations, we choose $\mathcal{V} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ and can write

$$\boxed{[T]_{\mathcal{V}}^{\mathcal{V}} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{pmatrix}}$$

2. *(20 pts)* Compute the coefficients c_1, c_2, c_3 in the equation below WITHOUT performing a row reduction or matrix inversion. (Be sure to explain your reasoning!)

$$\underbrace{\begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix}}_{\vec{v}} = c_1 \underbrace{\begin{pmatrix} -6/7 \\ 2/7 \\ 3/7 \end{pmatrix}}_{\vec{v}_1} + c_2 \underbrace{\begin{pmatrix} 2/7 \\ -3/7 \\ 6/7 \end{pmatrix}}_{\vec{v}_2} + c_3 \underbrace{\begin{pmatrix} 3/7 \\ 6/7 \\ 2/7 \end{pmatrix}}_{\vec{v}_3}$$

First, note that $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are all unit vectors, and are all orthogonal, so that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is an orthonormal basis.

We can therefore compute the coordinates c_1, c_2, c_3 as projections:

$$c_1 = \text{proj}_{\vec{v}_1}(\vec{v}) = \vec{v} \cdot \vec{v}_1 = \boxed{-23/7}$$

$$c_2 = \text{proj}_{\vec{v}_2}(\vec{v}) = \vec{v} \cdot \vec{v}_2 = \boxed{10/7}$$

$$c_3 = \text{proj}_{\vec{v}_3}(\vec{v}) = \vec{v} \cdot \vec{v}_3 = \boxed{29/7}$$

3. *(20 pts)* Show that the Hermitian dot product, defined by

$$\langle \vec{v}, \vec{w} \rangle_H = \sum v_i \overline{w_i} = \vec{v}^T \overline{\vec{w}}$$

has the property that

$$\langle \vec{v}, \vec{v} \rangle_H \geq 0, \text{ with equality iff } \vec{v} = \vec{0}$$

$$\langle \vec{v}, \vec{v} \rangle_H = \sum_k v_k \overline{v_k} = \sum (a_k + i b_k)(a_k - i b_k)$$

$$= \sum (a_k^2 + b_k^2)$$

This is a sum of squares of real numbers, which is zero only if all of those real numbers are zero.

So $a_k = 0$, $b_k = 0$, and thus $v_k = 0 + i0 = 0$.

This gives us $\vec{v} = \vec{0}$.

4. *(20 pts)* The matrix A has Jordan form

$$J = \begin{pmatrix} \boxed{\begin{matrix}5 & 1\\ 0 & 5\end{matrix}} & \begin{matrix}0 & 0\\ 0 & 0\end{matrix} \\ \begin{matrix}0 & 0\\ 0 & 0\end{matrix} & \boxed{\begin{matrix}5 & 1\\ 0 & 5\end{matrix}} \end{pmatrix}$$

and corresponding Jordan basis $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$, with

$$\vec{v}_1 = \begin{bmatrix}2\\4\\3\\1\end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix}3\\5\\1\\7\end{bmatrix} \quad \vec{v}_3 = \begin{bmatrix}0\\1\\1\\2\end{bmatrix} \quad \vec{v}_4 = \begin{bmatrix}2\\1\\5\\6\end{bmatrix}$$

Find a fundamental set of solutions to the system of equations $\vec{y}\,' = A\vec{y}$.

$\{\vec{v}_1, \vec{v}_2\}$ correspond to columns 1, 2, and thus the first Jordan block,

$\{\vec{v}_3, \vec{v}_4\}$ - - - - - - - - 3, 4 , - - - - - second - - - -

Using the formulas established in class, we then have

$$e^{xA}\vec{v}_1 = e^{5x}\vec{v}_1$$

$$e^{xA}\vec{v}_2 = e^{5x}(\vec{v}_2 + x\vec{v}_1)$$

$$e^{xA}\vec{v}_3 = e^{5x}\vec{v}_3$$

$$e^{xA}\vec{v}_4 = e^{5x}(\vec{v}_4 + x\vec{v}_3)$$

which form a fundamental set of solutions

$$\boxed{\left\{ e^{5x}\begin{pmatrix}2\\4\\3\\1\end{pmatrix},\ e^{5x}\begin{pmatrix}3+2x\\5+4x\\1+3x\\7+1x\end{pmatrix},\ e^{5x}\begin{pmatrix}0\\1\\1\\2\end{pmatrix},\ e^{5x}\begin{pmatrix}2\\1+1x\\5+1x\\6+2x\end{pmatrix} \right\}}$$

5. *(20 pts)* Find a Jordan basis and the Jordan canonical form for the matrix

$$A = \begin{pmatrix} 15 & 24 & -40 \\ -1 & 3 & 4 \\ 2 & 5 & -3 \end{pmatrix}$$

given that the characteristic polynomial of A is $p(\lambda) = (\lambda - 5)^3$, and

$$\left(\underbrace{\begin{matrix} 10 & 24 & -40 \\ -1 & -2 & 4 \\ 2 & 5 & -8 \end{matrix}}_{M} \middle| \underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{I}\right)$$

is row equivalent to

$$\left(\underbrace{\begin{matrix} 1 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{matrix}}_{R} \middle| \underbrace{\begin{matrix} 0 & -5 & -2 \\ 0 & 2 & 1 \\ 1 & 2 & -4 \end{matrix}}_{E}\right)$$

The given $p(\lambda)$ tells us that 5 is the only eigenvalue. We note that $M = A - 5I$, and R is in rref, so E is the row reduction matrix.

To find the eigenvector(s): We solve $(A-5I)\vec{v}_1 = M\vec{v}_1 = \vec{0}$ by

$$R\vec{v}_1 = E\vec{0}$$

or

$$\left(\begin{array}{ccc|c} 1 & 0 & -4 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Choosing $z = 1$, we get $\vec{v}_1 = \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}$

This is only 1 eigenvector, so there is only 1 basic Jordan block in this eigenvalue block. So the Jordan form is

$$\boxed{J = \begin{pmatrix} 5 & 1 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{pmatrix}}$$

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To find $\vec{v}_2$, we solve $M\vec{v}_2 = \vec{v}_1$

by $R\vec{v}_2 = E\vec{v}_1$

or
$$\left(\begin{array}{ccc|c} 1 & 0 & -4 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Choosing $z=0$, we get $\vec{v}_2 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$

To find $\vec{v}_3$, we solve $M\vec{v}_3 = \vec{v}_2$

by $R\vec{v}_3 = E\vec{v}_2$

or
$$\left(\begin{array}{ccc|c} 1 & 0 & -4 & -5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Choosing $z=0$, we get $\vec{v}_3 = \begin{pmatrix} -5 \\ 2 \\ 0 \end{pmatrix}$

So a Jordan basis is

$$\boxed{\left\{ \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -5 \\ 2 \\ 0 \end{pmatrix} \right\}}$$