EXAM 2

Math 107, 2011-2012 Spring, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.
All answers must be simplified. All of the policies and guidelines
on the class webpages are in effect on this exam.

Good luck!

Name Solutions

Rec: Number________ TA________ Day/Time________

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ________

1. ______
2. ______
3. ______
4. ______
5. ______
6. ______

Total Score ______ (/100 points)

1. *(20 pts)* In this problem we consider the vector space P_3 of all polynomials of degree less than or equal to 3. Find a (convenient) basis for P_3, show that it is a basis, and use this conclusion to compute $\dim(P_3)$. (Note, this means that you cannot make any assumptions about the dimension of P_3 while confirming your basis.)

Claim: $\{1, x, x^2, x^3\}$ is a basis for P_3

Independence: $W(x) = \det \begin{pmatrix} 1 & x & x^2 & x^3 \\ 0 & 1 & 2x & 3x^2 \\ 0 & 0 & 2 & 6x \\ 0 & 0 & 0 & 6 \end{pmatrix} = 12 \neq 0$

$W(x)$ is not identically zero, so we can conclude this collection of functions is independent.

Span: Every vector in P_3 is of the form

$$f = ax^3 + bx^2 + cx + d$$
$$= a(x^3) + b(x^2) + c(x) + d(1)$$

which is clearly a linear combination of the vectors in $\{1, x, x^2, x^3\}$. So this collection spans P_3

So $\{1, x, x^2, x^3\}$ is a basis.

There are four elements in this basis, so

$$\boxed{\dim(P_3) = 4}$$

2. *(15 pts)* By non-standard methods, your friend Bob has found four valid solutions to the differential equation

$$y'''' - 8y''' - 4y'' + 5y' + 2y = 0$$

Bob is now interested in the question:

Q: Are these four functions linearly independent or linearly dependent?

He has tried to compute the complete Wronskian, but he found the algebra to be too inconvenient to work out completely. However, he has been able to compute (correctly) that the value of the Wronskian at $x = 1$ is $w(1) = 0$. Based on this, he makes the following assertions:

(a) If $w(1)$ had turned out to be nonzero, that would have proved that the four functions were independent.

(b) The fact that $w(1) = 0$ does not demonstrate that the four functions are independent.

(c) The fact that $w(1) = 0$ does not demonstrate anything else relevant to the question (Q), so further calculations are necessary.

Identify each of the above three assertions as true or false; for each assertion that you identify as false, explain why Bob is wrong.

a) True.

b) True

c) False. Because these functions are solutions to a homogeneous linear DE satisfying the conditions of the existence/uniqueness theorem, and the number of functions is the same as the order of the DE, a theorem from class tells us that if the Wronskian is ever zero, then the functions must be dependent.

This does demonstrate the answer to the question (Q), and no further calculations are necessary.

3. *(20 pts)* Find a fundamental set of real solutions to the differential equation below:

$$y'''' - 4y''' + 27y = 0$$

(Hint: One real solution is $e^{-x}\sin(\sqrt{2}x)$.)

The given solution tells us that $(-1+\sqrt{2}\,i)$ and $(-1-\sqrt{2}\,i)$ are roots, and thus that a factor of $p(\lambda)$ is

$$\left(\lambda-(-1+\sqrt{2}i)\right)\left(\lambda-(-1-\sqrt{2}i)\right)$$

$$= (\lambda^2+2\lambda+3)$$

Dividing into $p(\lambda)$, we get

$$\begin{array}{r|l} & \lambda^2 - 6\lambda + 9 \\ \lambda^2+2\lambda+3 & \lambda^4 - 4\lambda^3 + 0\lambda^2 + 0\lambda + 27 \\ & \underline{\lambda^4 + 2\lambda^3 + 3\lambda^2} \\ & -6\lambda^3 - 3\lambda^2 + 0\lambda + 27 \\ & \underline{-6\lambda^3 - 12\lambda^2 - 18\lambda} \\ & 9\lambda^2 + 18\lambda + 27 \\ & \underline{9\lambda^2 + 18\lambda + 27} \\ & 0 \end{array}$$

Then $p(\lambda) = (\lambda^2+2\lambda+3)(\lambda^2-6\lambda+9)$

$$= \left(\lambda-(-1+\sqrt{2}i)\right)\left(\lambda-(-1-\sqrt{2}i)\right)(\lambda-3)^2$$

which as roots; $-1+\sqrt{2}\,i$, $-1-\sqrt{2}\,i$, 3 multiplicity $= 2$

Thus we have 4 solutions;

$$\{e^{-x}\sin(\sqrt{2}x),\ e^{-x}\cos(\sqrt{2}x),\ e^{3x},\ xe^{3x}\}$$

which are independent by the result from class, and thus form a fundamental set of (real) solutions.

4. *(15 pts)* Write down the form (but do not evaluate the constants!) for a particular solution to the differential equation below:

$$y''' - y'' + 3y' + 5y = x^2 e^x \cos(2x)$$

$p(\lambda) = \lambda^3 - \lambda^2 + 3\lambda + 5$ $r = 1+2i$

If r were a root, so would be $\bar{r} = 1-2i$, and thus a factor of $p(\lambda)$ would be

$$(\lambda - (1+2i))(\lambda - (1-2i)) = \lambda^2 - 2\lambda + 5$$

$$\begin{array}{r|l} & \lambda + 1 \\ \hline \lambda^2 - 2\lambda + 5 & \lambda^3 - \lambda^2 + 3\lambda + 5 \\ & \lambda^3 - 2\lambda^2 + 5\lambda \\ \hline & \lambda^2 - 2\lambda + 5 \\ & \lambda^2 - 2\lambda + 5 \\ \hline & 0 \end{array}$$

So we have

$$p(\lambda) = (\lambda - (1+2i))(\lambda - (1-2i))(\lambda - (-1))$$

and thus $r = 1+2i$ is a root of p of multiplicity 1.

By the theorem then, the form of the particular solution is

$$\boxed{\begin{aligned} y_p = {} & x(c_2x^2 + c_1x + c_0)e^x \cos(2x) \\ & + x(d_2x^2 + d_1x + d_0)e^x \sin(2x) \end{aligned}}$$

5. *(15 pts)* A mass on a spring in a frictionless medium is moving with position given by

$$u(t) = \cos(168t) - \cos(162t)$$

Use the angle addition formulas below to write this function as a product of sinusoidal waves.

$$\begin{aligned} \cos(a+b) &= \cos(a)\cos(b) - \sin(a)\sin(b) \\ \sin(a+b) &= \sin(a)\cos(b) + \cos(a)\sin(b) \\ \Rightarrow \cos(a-b) &= \cos(a)\cos(b) + \sin(a)\sin(b) \end{aligned}$$

So $\cos(a+b) - \cos(a-b) = -2\sin(a)\sin(b)$

To fit this identity to the above function, we choose

$$\left.\begin{aligned} a+b &= 168t \\ a-b &= 162t \end{aligned}\right\} \Longrightarrow a = 165t,\ b = 3t$$

So we get

$$\begin{aligned} u(t) &= \cos(168t) - \cos(162t) \\ &= \boxed{-2\sin(165t)\sin(3t)} \end{aligned}$$

6. *(15 pts)* The linear transformation $D : C^\infty \to C^\infty$ is computed by $D(f) = f'$, the linear transformation $T_a : C^\infty \to C^\infty$ (where a is a constant) is computed by $T_a(f) = af$, and the linear transformation $T_g : C^\infty \to C^\infty$ (where $g(x) = e^x$) is computed by $T_g(f) = gf = e^x f$

Prove your answer to each of the questions below:

(a) Do D and T_a commute (that is, does $DT_a = T_a D$)?

$$(DT_a)(f) = D(T_a(f)) = D(af) = (af)' = af'$$

$$(T_a D)(f) = T_a(D(f)) = T_a(f') = a(f') = af'$$

These are equal for all f, so $DT_a = T_a D$, and thus D and T_a <u>commute</u>.

(b) Do D and T_g commute?

$$(DT_g)(f) = D(T_g(f)) = D(gf) = (gf)' = gf' + g'f$$

$$(T_g D)(f) = T_g(D(f)) = T_g(f') = g(f') = gf'$$

These are not equal for $f \neq 0$. So $DT_g \neq T_g D$, and thus D and T_g <u>do not commute</u>.

(c) Do T_a and T_g commute?

$$(T_a T_g)(f) = T_a(T_g(f)) = T_a(gf) = agf$$

$$(T_g T_a)(f) = T_g(T_a(f)) = T_g(af) = g(af) = agf$$

These are equal for all f, so $T_a T_g = T_g T_a$, and thus T_a and T_g <u>commute</u>.