EXAM 1

Math 107, 2011-2012 Spring, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.
All answers must be simplified. All of the policies and guidelines
on the class webpages are in effect on this exam.

Good luck!

Name Solutions

ID number____________________

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ____________________

1. ________

2. ________

3. ________

4. ________

5. ________

6. ________

Total Score ________ (/100 points)

1. *(15 pts)* Find the unique matrix A that satisfies the equation below.

$$\left(\quad A \quad \right) \underbrace{\begin{pmatrix} 3 & 5 & 1 & -1 \\ 2 & 0 & 0 & 5 \\ 7 & 18 & 21 & 2 \end{pmatrix}}_{B} = \begin{pmatrix} 6 & 10 & 2 & -2 \\ 9 & 18 & 21 & 7 \\ 4 & 0 & 0 & 10 \end{pmatrix}$$

We observe:

- 1st row of product is 2 times 1st row of B
- 2nd . . . . 2nd row of B plus 3rd row of B
- 3rd . . . . 2 times 2nd row of B.

Using the "linear combination of rows" view of matrix products,

we conclude

$$\boxed{\underbrace{\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 0 \end{pmatrix}}_{A}} \begin{pmatrix} 3 & 5 & 1 & -1 \\ 2 & 0 & 0 & 5 \\ 7 & 18 & 21 & 2 \end{pmatrix} = \begin{pmatrix} 6 & 10 & 2 & -2 \\ 9 & 18 & 21 & 7 \\ 4 & 0 & 0 & 10 \end{pmatrix}$$

2. *(15 pts)* Suppose your friend Bob tells you that the system $A\vec{x} = \vec{b}$ has at least one solution for every vector $\vec{b}$. Suppose that Bob also tells you he has found two distinct solutions to the system when $\vec{b} = \vec{b}_1$.

Use observations about the pivots in rref(A) and rref(A^T) to show that A^T cannot have the existence property.

rows

Say A is $m \times n$ columns

Bob's first statement tells us $\text{rank}(A) = m$

Bob's second statement tells us $\text{rank}(A) < n$

So we know $m < n$.

rows columns

A^T is $n \times m$, and thus must have more rows than columns. So we get

$\text{rank}(A^T) \leq m < n$

This means that rref(A^T) must have at least one row with no pivot.

So A^T cannot have the existence property.

3. *(15 pts)*

(a) Use a row reduction to compute the inverse of the matrix

$$A = \begin{pmatrix} 3 & 8 \\ 1 & 2 \end{pmatrix}$$

$$\left(\begin{array}{cc|cc} 3 & 8 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 3 & 8 & 1 & 0 \end{array}\right) \begin{array}{l} 2 \\ 1 \end{array}$$

$$\left(\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 0 & 2 & 1 & -3 \end{array}\right) \begin{array}{l} 1 \\ 2 - 3\,1 \end{array}$$

$$\left(\begin{array}{cc|cc} 1 & 0 & -1 & 4 \\ 0 & 2 & 1 & -3 \end{array}\right) \begin{array}{l} 1 - 2 \\ 2 \end{array}$$

$$\left(\begin{array}{cc|cc} 1 & 0 & -1 & 4 \\ 0 & 1 & 1/2 & -3/2 \end{array}\right) \begin{array}{l} 1 \\ \frac{1}{2}\,2 \end{array}$$

$\text{rref}(A) = I$

and

$$\boxed{A^{-1} = \begin{pmatrix} -1 & 4 \\ 1/2 & -3/2 \end{pmatrix}}$$

(b) Explicitly use the row reduction above to compute the determinant of A.

From the reduction above, we have

$$1 = \det I = -\tfrac{1}{2} \det A$$

So $\boxed{\det A = -2}$

(c) By any appropriate means, find all values of c for which the matrix below is not invertible.

$$B = \begin{pmatrix} 3 & c \\ 1 & 2 \end{pmatrix}$$

B not invtbl $\iff \det B = 0 \iff 6 - c = 0$

So $\boxed{c = 6}$ is the only such value of c.

4. *(20 pts)* The nonsingular 3×3 matrix M is brought to its reduced row echelon form by way of the following ordered sequence of elementary row operations:

   1. The first row is added to the second row.
   2. The second row is multiplied by 7.
   3. Five times the third row is added to the first row.
   4. The second and third rows are switched.

   Write the matrix M as a product of elementary matrices.

We can represent the row reduction as

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}}_{E_4} \underbrace{\begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{E_3} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{E_2} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{E_1} M = I$$

This gives us

$$M = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1}$$

$$= \boxed{\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/7 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}}$$

5. *(20 pts)* The $n \times n$ matrix A has rows $A_1, \ldots, A_n$. The matrix obtained by crossing out the ith row and jth column of A is M_{ij}.

The matrix P is obtained by replacing the ith row of A by $V = (v_1 \ \cdots \ v_n)$. The matrix Q is obtained by replacing the ith row of A by $W = (w_1 \ \cdots \ w_n)$. The matrix R is obtained by replacing the ith row of A by $aV + bW$:

$$R = \begin{bmatrix} A_1 \\ \vdots \\ aV + bW \\ \vdots \\ A_n \end{bmatrix} \quad P = \begin{bmatrix} A_1 \\ \vdots \\ V \\ \vdots \\ A_n \end{bmatrix} \quad Q = \begin{bmatrix} A_1 \\ \vdots \\ W \\ \vdots \\ A_n \end{bmatrix}$$

Using this notation then, the ith row cofactor expansion for the determinant of P is

$$\det P = \sum_{j=1}^{n} v_j (-1)^{i+j} \det(M_{ij})$$

Prove the multilinearity of determinant in rows by showing that

$$\det R = a \det P + b \det Q$$

Using the notation above, and the ith row cofactor expansion:

$$\det R = \sum_{j=1}^{n} (aV + bW)_j (-1)^{i+j} \det(M_{ij})$$

$$= \sum_{j=1}^{n} (av_j + bw_j)(-1)^{i+j} \det(M_{ij})$$

$$= a \sum_{j=1}^{n} v_j (-1)^{i+j} \det(M_{ij}) + b \sum_{j=1}^{n} w_j (-1)^{i+j} \det(M_{ij})$$

$$= a \det P + b \det Q$$

6. *(15 pts)* Determine if the collection of vectors below is linearly independent or linearly dependent.

$$\left\{ \begin{bmatrix} 2 \\ 3 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 9 \\ 7 \\ 3 \\ -2 \end{bmatrix} \right\}$$

$$A = \begin{pmatrix} 2 & 5 & 9 \\ 3 & 1 & 7 \\ 1 & 1 & 3 \\ -2 & 2 & -2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 3 \\ 3 & 1 & 7 \\ 2 & 5 & 9 \\ -2 & 2 & -2 \end{pmatrix} \begin{matrix} 3 \\ 2 \\ 1 \\ 4 \end{matrix}$$

$$\begin{pmatrix} 1 & 1 & 3 \\ 0 & -2 & -2 \\ 0 & 3 & 3 \\ 0 & 4 & 4 \end{pmatrix} \begin{matrix} 1 \\ 2 - 3(1) \\ 3 - 2(1) \\ 4 + 2(1) \end{matrix}$$

$$\begin{pmatrix} 1 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 3 & 3 \\ 0 & 4 & 4 \end{pmatrix} \begin{matrix} 1 \\ -\frac{1}{2}(2) \\ 3 \\ 4 \end{matrix}$$

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{matrix} 1 - 2 \\ 2 \\ 3 - 3(2) \\ 4 - 4(2) \end{matrix}$$

rref (A) does <u>not</u> have a pivot in every column

So $A\vec{c} = \vec{0}$ does not have unique solutions.

So $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$ does not have only the trivial solution.

So this collection of vectors is $\boxed{\text{linearly dependent.}}$