EXAM 3

Math 107, 2011-2012 Fall, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.
All answers must be simplified. All of the policies and guidelines
on the class webpages are in effect on this exam.

Good luck!

Name Solutions

ID number____________________

"I have adhered to the Duke Community
Standard in completing this
examination."

Signature: ____________________

1. __________

2. __________

3. __________

4. __________

5. __________

Total Score __________ (/100 points)

1. ([redacted] *pts)* Compute all of the eigenvalues and eigenvectors for the matrix

$$\begin{pmatrix} 11 & -18 \\ 3 & -4 \end{pmatrix}$$

$$A-\lambda I = \begin{pmatrix} 11-\lambda & -18 \\ 3 & -4-\lambda \end{pmatrix}$$

$$p(\lambda) = \det(A-\lambda I) = (11-\lambda)(-4-\lambda) + 54 = -44 - 7\lambda + \lambda^2 + 54$$

$$= \lambda^2 - 7\lambda + 10 = (\lambda-2)(\lambda-5)$$

$$\boxed{\text{Eigenvalues are } 2,\ 5}$$

<u>For $\lambda = 2$:</u>

$$A-\lambda I = \begin{pmatrix} 9 & -18 \\ 3 & -6 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix} \begin{array}{l} 2/3 \\ 1 - 3\,2 \end{array} \Rightarrow x = 2y \Rightarrow \vec{v}_1 = \boxed{k\begin{pmatrix} 2 \\ 1 \end{pmatrix}}$$

<u>For $\lambda = 5$:</u>

$$A-\lambda I = \begin{pmatrix} 6 & -18 \\ 3 & -9 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -3 \\ 0 & 0 \end{pmatrix} \begin{array}{l} 2/3 \\ 1 - 2\,2 \end{array} \Rightarrow x = 3y \Rightarrow \boxed{\vec{v}_2 = k\begin{pmatrix} 3 \\ 1 \end{pmatrix}}$$

2.[redacted] ([redacted] *pts)* Determine if the following vector-valued functions are linearly independent or linearly dependent:

$$\vec{y}_1(x) = \begin{pmatrix} e^x \\ 3e^x \\ -2e^x \end{pmatrix} \quad \text{and} \quad \vec{y}_2(x) = \begin{pmatrix} 2e^{4x} \\ -e^{4x} \\ e^{4x} \end{pmatrix} \quad \text{and} \quad \vec{y}_3(x) = \begin{pmatrix} e^{3x} \\ e^{3x} \\ 7e^{3x} \end{pmatrix}$$

At $x=0$, the Wronskian of these functions is

$$W(0) = \det \begin{pmatrix} 1 & 2 & 1 \\ 3 & -1 & 1 \\ -2 & 1 & 7 \end{pmatrix}$$

$$= (1)(-8) - (2)(23) + (1)(1)$$

$$= -53 \neq 0$$

So these functions are linearly independent

3. (*20 pts*) Find a fundamental set of solutions to the system $\vec{y}' = A\vec{y}$, where

$$A = \begin{pmatrix} 5 & -3 & 1 \\ 3 & -2 & 2 \\ 3 & -5 & 5 \end{pmatrix}$$

You may use the fact that the Jordan form for A and the corresponding Jordan basis are given by

$$J = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad \text{and} \quad \left\{ \underbrace{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}_{\vec{v}_1}, \underbrace{\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}}_{\vec{v}_2}, \underbrace{\begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix}}_{\vec{v}_3} \right\}$$

$$\left.\begin{aligned} e^{xA}\vec{v}_1 &= e^{3x}(\vec{v}_1) \\ e^{xA}\vec{v}_2 &= e^{3x}(\vec{v}_2 + x\vec{v}_1) \\ e^{xA}\vec{v}_3 &= e^{2x}(\vec{v}_3) \end{aligned}\right\} \text{fundamental set}$$

We can plug in these vectors to write the fundamental set as

$$\boxed{\left\{ e^{3x}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \; e^{3x}\begin{pmatrix} 3+x \\ 2+x \\ 1+x \end{pmatrix}, \; e^{2x}\begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix} \right\}}$$

4. (*20 pts*) Find the Jordan form and a Jordan basis for the matrix

$$A = \begin{pmatrix} 14 & -16 \\ 9 & -10 \end{pmatrix}$$

$$p(\lambda) = \det(A - \lambda I) = (14-\lambda)(-10-\lambda) + 144$$

$$= -140 - 4\lambda + \lambda^2 + 144 \quad = \lambda^2 - 4\lambda + 4 = (\lambda-2)^2$$

Only one eigenvalue, $\lambda = 2$, and thus only 1 eigenvalue block.

$$A - \lambda I = \begin{pmatrix} 12 & -16 \\ 9 & -12 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -4/3 \\ 0 & 0 \end{pmatrix} \begin{matrix} ①/12 \\ ② - \frac{3}{4}① \end{matrix} \quad \Rightarrow x = \frac{4}{3}y \quad \Rightarrow \vec{v}_1 = k\begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

is the only eigenvector.

This tells us the Jordan form must be

$$\boxed{J = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}}$$

To find the next basis vector, $\vec{v}_2$, we solve

$$(A - \lambda I)\vec{v}_2 = \vec{v}_1$$

$$\begin{pmatrix} 12 & -16 \\ 9 & -12 \end{pmatrix} \vec{v}_2 = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 12 & -16 & 4 \\ 9 & -12 & 3 \end{array}\right) \begin{matrix} ①/12 \\ ② - \frac{3}{4}① \end{matrix}$$

$$\left(\begin{array}{cc|c} 1 & -\frac{4}{3} & 1/3 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow x = \frac{4}{3}y + \frac{1}{3}$$

we choose $y = 2$, and get $x = 3$

A Jordan basis then is

$$\{\vec{v}_1, \vec{v}_2\} = \boxed{\left\{ \begin{pmatrix} 4 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \end{pmatrix} \right\}}$$

5. (*20 pts*) Find a particular solution to the system of equations

$$\begin{aligned} y_1' &= y_2 + x + 1 \\ y_2' &= y_1 + 2x - 1 \end{aligned}$$

$$\vec{y}' = A\vec{y} + \vec{v}_1 + \vec{v}_2 x$$

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

We guess $\vec{y}_p = \vec{a} + \vec{b}x$. Plugging in, we get

$$\vec{b} = A(\vec{a} + \vec{b}x) + \vec{v}_1 + \vec{v}_2 x$$

$$\vec{0} = \underbrace{\left(A\vec{a} + \vec{v}_1 - \vec{b}\right)}_{=\vec{0}} + \underbrace{\left(A\vec{b} + \vec{v}_2\right)}_{=\vec{0}} x$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\begin{aligned} \vec{a} &= A^{-1}(\vec{b} - \vec{v}_1) \\ &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -3 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ -3 \end{pmatrix} \end{aligned} \qquad \begin{aligned} \vec{b} &= A^{-1}(-\vec{v}_2) \\ &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -1 \\ -2 \end{pmatrix} \\ &= \begin{pmatrix} -2 \\ -1 \end{pmatrix} \end{aligned}$$

Then

$$\begin{aligned} \vec{y}_p &= \vec{a} + \vec{b}x \\ &= \begin{pmatrix} 0 \\ -3 \end{pmatrix} + \begin{pmatrix} -2 \\ -1 \end{pmatrix} x \\ &= \begin{pmatrix} -2x \\ -3 - x \end{pmatrix} \end{aligned}$$