# EXAM 2

Math 107, 2011-2012 Fall, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.
All answers must be simplified. All of the policies and guidelines
on the class webpages are in effect on this exam.

Good luck!

Name Solutions

ID number____________________

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ____________________

1. __________
2. __________
3. __________
4. __________
5. __________
6. __________
7. __________

Total Score __________ (/100 points)

1. *(10 pts)* Decide if the following collection of functions is linearly independent or linearly dependent.

$$\{x, \sin x, \cos x, \sin(x^2)\}$$

$f_1 = x$ $\quad f_1' = 1$ $\quad f_1'' = 0$ $\quad f_1''' = 0$

$f_2 = \sin x$ $\quad f_2' = \cos x$ $\quad f_2'' = -\sin x$ $\quad f_2''' = -\cos x$

$f_3 = \cos x$ $\quad f_3' = -\sin x$ $\quad f_3'' = -\cos x$ $\quad f_3''' = \sin x$

$f_4 = \sin(x^2)$

$f_4' = 2x\cos(x^2)$

$f_4'' = 2\cos(x^2) - 4x^2\sin(x^2)$

$f_4''' = -4x\sin(x^2) - 8x\sin(x^2) - 8x^3\cos(x^2)$

$= -12x\sin(x^2) - 8x^3\cos(x^2)$

$$W(0) = \det\begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 2 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

expand along top row

$$= \det\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & -1 & 0 \end{pmatrix}$$

expand along left column

$$= \det\begin{pmatrix} 0 & 2 \\ -1 & 0 \end{pmatrix}$$

$$= 2 \neq 0$$

So this collection is linearly independent.

2. *(15 pts)* Consider the differential equation

$$L(y) = e^x y'' + (\sin x)y' - xy = 0$$

(a). Identify the specific features of this equation that allow you to conclude that initial value problems with this differential equation have unique solutions.

— linear

— coefficient functions are continuous, as is RHS

— leading coefficient function is never 0.

(b) Suppose we know solutions $f$ and $g$ to the above differential equation, with $f(0) = 1$, $f'(0) = 3$, $g(0) = 2$, $g'(0) = -1$. Find a solution to the initial value problem

$$L(y) = 0, \quad y(0) = 5,\ y'(0) = 1$$

$f$ has initial values $\begin{pmatrix} f(0) \\ f'(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$g$ has initial values $\begin{pmatrix} g(0) \\ g'(0) \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

$c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$ has (!) sol $c_1 = 1, c_2 = 2$

Then $y = 1f + 2g$ has

$L(y) = L(f+2g) = L(f) + 2L(g) = 0$

and

$y(0) = f(0) + 2g(0) = 5$

$y'(0) = f'(0) + 2g'(0) = 1$

3. *(15 pts)* Find a fundamental set of real solutions to the differential equation

$p(\lambda) = \lambda^4 + 3\lambda^3 + 4\lambda^2 + 3\lambda + 1$

$$y'''' + 3y''' + 4y'' + 3y' + y = 0$$

Possible rat'l roots : $1, -1$

$p(1) = 12$

$p(-1) = 0 \leftarrow$ $-1$ is a root so $(\lambda+1)$ is a factor

$$\begin{array}{r|l} & \lambda^3 + 2\lambda^2 + 2\lambda + 1 \\ \lambda+1 & \lambda^4 + 3\lambda^3 + 4\lambda^2 + 3\lambda + 1 \\ & \underline{\lambda^4 + \lambda^3} \\ & 2\lambda^3 + 4\lambda^2 + 3\lambda + 1 \\ & \underline{2\lambda^3 + 2\lambda^2} \\ & 2\lambda^2 + 3\lambda + 1 \\ & \underline{2\lambda^2 + 2\lambda} \\ & \lambda + 1 \\ & \underline{\lambda + 1} \\ & 0 \end{array}$$

$$p(\lambda) = (\lambda+1)\underbrace{(\lambda^3 + 2\lambda^2 + 2\lambda + 1)}_{g_1(\lambda)}$$

$g_1$ has possible rat'l roots : $1, -1$

$g_1(-1) = 0 \leftarrow$ $-1$ is a root, $(\lambda+1)$ is a factor

$$\begin{array}{r|l} & \lambda^2 + \lambda + 1 \\ \lambda+1 & \lambda^3 + 2\lambda^2 + 2\lambda + 1 \\ & \underline{\lambda^3 + \lambda^2} \\ & \lambda^2 + 2\lambda + 1 \\ & \underline{\lambda^2 + \lambda} \\ & \lambda + 1 \\ & \underline{\lambda + 1} \\ & 0 \end{array}$$

$$p(\lambda) = (\lambda+1)^2 \underbrace{(\lambda^2 + \lambda + 1)}_{g_2}$$

$g_2$ has roots $\dfrac{-1 \pm \sqrt{-3}}{2}$

$$= -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

So a real fundamental set of solutions is

$$\left\{ e^{-x},\ xe^{-x},\ e^{-\frac{1}{2}x}\cos\left(\frac{\sqrt{3}}{2}x\right),\ e^{-\frac{1}{2}}\sin\left(\frac{\sqrt{3}}{2}x\right) \right\}$$

(independent, as discussed in class)

4. *(15 pts)* Find a particular solution to the differential equation

$\lambda^3+2\lambda^2-\lambda+5=p(\lambda) \rightarrow$ $$y''' + 2y'' - y' + 5y = \sin x$$ $\leftarrow r = 0+1i = i$

$r$ is <u>not</u> a root of $p(\lambda)$

So we guess $y = a\sin x + b\cos x$

then $y' = a\cos x - b\sin x$

$y'' = -a\sin x - b\cos x$

$y''' = -a\cos x + b\sin x$

The equation then becomes

$$(b-2a+b+5a)\sin x + (-a-2b-a+5b)\cos x = \sin x$$

$$\Longrightarrow \quad \begin{aligned} 3a+2b &= 1 \\ -2a+3b &= 0 \end{aligned} \quad \Longrightarrow \quad \begin{aligned} a &= \tfrac{3}{13} \\ b &= \tfrac{2}{13} \end{aligned}$$

So a particular solution is

$$Y_p = \frac{3}{13}\sin x + \frac{2}{13}\cos x$$

5. *(15 pts)* Prove that the function $T : C^0 \to \mathbb{R}^1$ defined by

$$T(f) = \int_0^1 f(x)e^x \, dx$$

is a linear transformation.

$$T(c_1 f_1 + c_2 f_2) = \int_0^1 (c_1 f_1 + c_2 f_2) e^x \, dx$$

$$= \int_0^1 c_1 f_1 e^x + c_2 f_2 e^x \, dx$$

$$= c_1 \int_0^1 f_1 e^x \, dx + c_2 \int_0^1 f_2 e^x \, dx$$

$$= c_1 T(f_1) + c_2 T(f_2)$$

6. *(15 pts)* Use algebra of linear transformations to show that any solution to the differential equation

$$L_1(y) = y^{[7]} + 5y^{[6]} - 3y^{[5]} + y^{[4]} - y''' + 7y'' + 2y' + 5y = 0$$

must also be a solution to the differential equation

$$L_2(y) = y^{[8]} + 6y^{[7]} + 2y^{[6]} - 2y^{[5]} + 0y^{[4]} + 6y''' + 9y'' + 7y' + 5y = 0$$

(Do NOT refer to fundamental sets of solutions for these equations.)
*(Hint:* $\underbrace{(\lambda^7+5\lambda^6-3\lambda^5+\lambda^4-\lambda^3+7\lambda^2+2\lambda+5)}_{g(\lambda)}(\lambda+1) = \underbrace{(\lambda^8+6\lambda^7+2\lambda^6-2\lambda^5+0\lambda^4+6\lambda^3+9\lambda^2+7\lambda+5)}_{p(\lambda)}$.*)*

If $y$ is a solution to $L_1(y) = 0$, then

$$L_1(y) = g(D)(y) = 0$$

Then

$$\begin{aligned} L_2(y) &= p(D)(y) \\ &= \big(g(D)(D+1)\big)(y) \\ &= \big((D+1)\,g(D)\big)(y) \\ &= (D+1)\big(g(D)(y)\big) \\ &= (D+1)(0) \\ &= 0 \end{aligned}$$

So $y$ is also a solution to $L_2(y) = 0$.

7. *(15 pts)* The linear transformation $T : P^4 \to P^4$ ($P^4$ is the vector space of polynomials of degree at most 4) is defined by

$$T(y) = y'' - 3y' + 2y$$

Compute $[T]_{\mathcal{V}}^{\mathcal{V}}$, where $\mathcal{V} = \{\underset{\substack{\uparrow \\ f_1}}{1}, \underset{\substack{\uparrow \\ f_2}}{x}, \underset{\substack{\uparrow \\ f_3}}{x^2}, \underset{\substack{\uparrow \\ f_4}}{x^3}, \underset{\substack{\uparrow \\ f_5}}{x^4}\}$.

$$T(f_1) = 2 = 2f_1 + 0f_2 + 0f_3 + 0f_4 + 0f_5$$

$$T(f_2) = -3 + 2x = -3f_1 + 2f_2 + 0f_3 + 0f_4 + 0f_5$$

$$T(f_3) = 2 - 6x + 2x^2 = 2f_1 - 6f_2 + 2f_3 + 0f_4 + 0f_5$$

$$T(f_4) = 6x - 9x^2 + 2x^3 = 0f_1 + 6f_2 - 9f_3 + 2f_4 + 0f_5$$

$$T(f_5) = 12x^2 - 12x^3 + 2x^4 = 0f_1 + 0f_2 + 12f_3 - 12f_4 + 2f_5$$

So
$$[T(f_1)]_{\mathcal{V}} = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad [T(f_2)]_{\mathcal{V}} = \begin{pmatrix} -3 \\ 2 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad [T(f_3)]_{\mathcal{V}} = \begin{pmatrix} 2 \\ -6 \\ 2 \\ 0 \\ 0 \end{pmatrix}$$

$$[T(f_4)]_{\mathcal{V}} = \begin{pmatrix} 0 \\ 6 \\ -9 \\ 2 \\ 0 \end{pmatrix} \quad [T(f_5)]_{\mathcal{V}} = \begin{pmatrix} 0 \\ 0 \\ 12 \\ -12 \\ 2 \end{pmatrix}$$

And thus

$$[T]_{\mathcal{V}}^{\mathcal{V}} = \begin{pmatrix} 2 & -3 & 2 & 0 & 0 \\ 0 & 2 & -6 & 6 & 0 \\ 0 & 0 & 2 & -9 & 12 \\ 0 & 0 & 0 & 2 & -12 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$