EXAM 1

Math 107, 2011-2012 Fall, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.
All answers must be simplified. All of the policies and guidelines
on the class webpages are in effect on this exam.

Good luck!

Name Solutions

ID number____________________

1. __________

2. __________

3. __________

4. __________

5. __________

6. __________

7. __________

"I have adhered to the Duke Community
Standard in completing this
examination."

Signature: ____________________

Total Score __________ (/100 points)

1. *(12 pts)* In this problem we consider the system below.

$$\begin{aligned} e^2x_1 + x_2 - 14x_3 + \tan(\pi/7)x_4 &= 0 \\ \sin(31)x_1 + \sin(e)x_2 + \pi^e x_3 + x_4 &= 0 \\ 13.2x_1 + \cos(e)x_2 + 2x_3 + 23x_4 &= 0 \end{aligned}$$

(a) Show that this system of equations must have at least one solution.

$\vec{x} = \vec{0}$ is clearly a solution.

(b) Show that this system of equations must have more than one solution.

The coefficient matrix has 4 columns but only 3 rows, so the rank is at most 3. Thus there is a free variable, which (along with part (a)) tells us there are infinitely many solutions.

2. *(13 pts)* Bob has a 4×6 matrix A. He would like to combine several row operations into one step, resulting in the following effects: add 4 times the second row to the third row, switch the first and fourth rows, and multiply the second row by 8.

What single matrix B could Bob multiply by A on the left to accomplish all of this?

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 8 & 0 & 0 \\ 0 & 4 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \vec{r}_1 \\ \vec{r}_2 \\ \vec{r}_3 \\ \vec{r}_4 \end{pmatrix} = \begin{pmatrix} \vec{r}_4 \\ 8\vec{r}_2 \\ 4\vec{r}_2 + \vec{r}_3 \\ \vec{r}_1 \end{pmatrix}$$

Interpreting matrix multiplication by rows of the product as linear combinations of the rows of the right matrix (A), we see that the desired effect is accomplished by

$$B = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 8 & 0 & 0 \\ 0 & 4 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

3. *(15 pts)* In this problem we consider these 4×4 matrices with row vectors indicated as below.

$$A = \begin{pmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vec{a}_3 \\ \vec{a}_4 \end{pmatrix} \quad L = \begin{pmatrix} \vec{a}_2 \\ \vec{a}_3 \\ \vec{a}_4 \\ \vec{a}_1 \end{pmatrix} \quad M = \begin{pmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vec{v} \\ \vec{a}_4 \end{pmatrix} \quad N = \begin{pmatrix} \vec{v} \\ \vec{a}_2 \\ \vec{a}_3 \\ \vec{w} \end{pmatrix}$$

Suppose we know that $\det A = 4$, and

$$\begin{aligned} \vec{v} &= 3\vec{a}_1 + 6\vec{a}_2 - 8\vec{a}_3 + 1\vec{a}_4 \\ \vec{w} &= 2\vec{a}_1 + 12\vec{a}_2 + 2\vec{a}_3 + 4\vec{a}_4 \end{aligned}$$

Compute $\det L$, $\det M$, and $\det N$. (Make sure to explain your reasoning clearly and completely!)

L is obtained by three row switches on A. So

$$\det L = (-1)^3 \det A = \boxed{-4}$$

By multilinearity on the 3rd row,

$$\det M = 3\det\begin{pmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vec{a}_1 \\ \vec{a}_4 \end{pmatrix} + 6\det\begin{pmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vec{a}_2 \\ \vec{a}_4 \end{pmatrix} - 8\det\begin{pmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vec{a}_3 \\ \vec{a}_4 \end{pmatrix} + 1\det\begin{pmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vec{a}_4 \\ \vec{a}_4 \end{pmatrix}$$

$$= 0 + 0 - 8(4) + 0 = \boxed{-32}$$

Using multilinearity on the 1st, 4th rows of N and eliminating terms with repeated rows, we get

$$\det N = \det\begin{pmatrix} 3\vec{a}_1 + 1\vec{a}_4 \\ \vec{a}_2 \\ \vec{a}_3 \\ 2\vec{a}_1 + 4\vec{a}_4 \end{pmatrix}$$

$$= \det\begin{pmatrix} 3\vec{a}_1 \\ \vec{a}_2 \\ \vec{a}_3 \\ 4\vec{a}_4 \end{pmatrix} + \det\begin{pmatrix} 1\vec{a}_4 \\ \vec{a}_2 \\ \vec{a}_3 \\ 2\vec{a}_1 \end{pmatrix}$$

$$= 12\det A - 2\det A = \boxed{40}$$

4. *(15 pts)* Find the inverse of the matrix A below.

$$A = \begin{pmatrix} 1 & 4 & 1 \\ 3 & 14 & -1 \\ 8 & 36 & 10 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 4 & 1 & 1 & 0 & 0 \\ 3 & 14 & -1 & 0 & 1 & 0 \\ 8 & 36 & 10 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 4 & 1 & 1 & 0 & 0 \\ 0 & 2 & -4 & -3 & 1 & 0 \\ 0 & 4 & 2 & -8 & 0 & 1 \end{array}\right) \begin{array}{l} ① \\ ② - 3① \\ ③ - 8① \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 9 & 7 & -2 & 0 \\ 0 & 1 & -2 & -3/2 & 1/2 & 0 \\ 0 & 0 & 10 & -2 & -2 & 1 \end{array}\right) \begin{array}{l} ① - 2② \\ ②/2 \\ ③ - 2② \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 9 & 7 & -2 & 0 \\ 0 & 1 & -2 & -3/2 & 1/2 & 0 \\ 0 & 0 & 1 & -1/5 & -1/5 & 1/10 \end{array}\right) \begin{array}{l} ① \\ ② \\ ③/10 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 44/5 & -1/5 & -9/10 \\ 0 & 1 & 0 & -19/10 & 1/10 & 1/5 \\ 0 & 0 & 1 & -1/5 & -1/5 & 1/10 \end{array}\right) \begin{array}{l} ① - 9③ \\ ② + 2③ \\ ③ \end{array}$$

rref$(A) = I$, so this is A^{-1}

5. *(15 pts)* The collection of vectors

$$\left\{ \begin{bmatrix} 1 \\ 3 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} 12 \\ 3 \\ 1 \\ -6 \end{bmatrix}, \begin{bmatrix} 28 \\ 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 2 \\ 5 \end{bmatrix} \right\}$$

is linearly independent. Determine if the collection of vectors

$$\left\{ \begin{bmatrix} 28 \\ 12 \\ 7 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 8 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ -6 \\ 5 \\ 4 \end{bmatrix} \right\}$$

is linearly independent or linearly dependent. (Make sure to explain your reasoning clearly and completely!)

The given information tells us that

$$\det \begin{pmatrix} 1 & 12 & 28 & 7 \\ 3 & 3 & 2 & 8 \\ -2 & 1 & -2 & 2 \\ 4 & -6 & 1 & 5 \end{pmatrix} \neq 0$$

Transposing does not change the determinant, so

$$\det \begin{pmatrix} 1 & 3 & -2 & 4 \\ 12 & 3 & 1 & -6 \\ 28 & 2 & -2 & 1 \\ 7 & 8 & 2 & 5 \end{pmatrix} \neq 0$$

Switching rows introduces factors of (-1), but then still

$$\det \begin{pmatrix} 28 & 2 & -2 & 1 \\ 12 & 3 & 1 & -6 \\ 7 & 8 & 2 & 5 \\ 1 & 3 & -2 & 4 \end{pmatrix} \neq 0$$

The columns of this matrix are thus independent, which is what was to be decided.

6. *(15 pts)* In this problem we consider the set S defined by

$$S = \left\{ f \in C^1(\mathbb{R}) \;\middle|\; f(3) = 0 \text{ and } f'(5) = 0 \right\}$$

Prove that S is a subpace of C^1.

Say f, g are in S. Then $f(3) = 0$, $f'(5) = 0$
$g(3) = 0$, $g'(5) = 0$

Then $(f+g)(3) = f(3) + g(3) = 0 + 0 = 0$
$(f+g)'(5) = f'(5) + g'(5) = 0 + 0 = 0$

So $f+g$ is in S.

So S is closed under addition.

Say f is in S. Then $f(3) = 0$, $f'(5) = 0$

Then $(cf)(3) = c f(3) = c \cdot 0 = 0$
$(cf)'(5) = c f'(5) = c \cdot 0 = 0$

So cf is in S.

So S is closed under scalar multiplication.

Therefore S is a subspace of C^1.

7. *(15 pts)* Consider the collection of functions

$$S = \left\{ \sin(x - \phi) \;\middle|\; \phi \in [0, 2\pi) \right\}$$

These functions are sine waves of amplitude 1 and period 2π, with different phase shifts ϕ. There are infinitely many functions in this collection, since ϕ can be any of infinitely many values.

The collection S is not a subspace of C^∞, but of course $V = \text{span}(S)$ (the set of all functions that are finite linear combinations of functions in S) is a subspace of C^∞.

Decide if V is finite dimensional or infinite dimensional; and if it is finite dimensional, compute the dimension of the subspace V. *(Hint: You can make use here of the angle addition formula* $\sin(a + b) = \sin a \cos b + \cos a \sin b$.*)*

$$\sin(x-\phi) = \sin x \cos(-\phi) + \cos x \sin(-\phi)$$
$$= (\cos\phi) \sin x + (-\sin\phi) \cos x$$

Thus every vector in S is a linear combination of $\{\sin x, \cos x\}$.

So every linear combination of vectors in S is also, and thus

$$V \subset \text{span}\{\sin x, \cos x\}$$

We know that vectors in V are not all multiples of each other, so $\dim V > 1$, and $\dim(\text{span}\{\sin x, \cos x\}) = 2$, so we must have

$$\boxed{\dim V = 2}$$