EXAM 3

Math 107, 2010-2011 Spring, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.
All answers must be simplified. All of the policies and guidelines
on the class webpages are in effect on this exam.

Good luck!

Name Solutions

ID number____________________

"I have adhered to the Duke Community
Standard in completing this
examination."

Signature: ____________________

1. __________
2. __________
3. __________
4. __________
5. __________

Total Score __________ (/100 points)

1. *(20 pts)* Find the matrix $A = [T]_{\mathcal{S}}^{\mathcal{S}}$, given the information below:

$$[T]_{\mathcal{V}}^{\mathcal{V}} = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\mathcal{V} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}, \quad [\vec{v}_1]_{\mathcal{S}} = \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix} \quad [\vec{v}_2]_{\mathcal{S}} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \quad [\vec{v}_3]_{\mathcal{S}} = \begin{bmatrix} 2 \\ 5 \\ -2 \end{bmatrix}$$

$$\begin{pmatrix} -9 & 8 & 11 \\ -2 & 2 & 3 \\ 8 & -7 & -10 \end{pmatrix} \begin{pmatrix} 1 & 3 & 2 \\ 4 & 2 & 5 \\ -2 & 1 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$[T]_{\mathcal{S}}^{\mathcal{S}} = [I]_{\mathcal{V}}^{\mathcal{S}} \, [T]_{\mathcal{V}}^{\mathcal{V}} \, [I]_{\mathcal{S}}^{\mathcal{V}}$$

$$= \underbrace{\begin{pmatrix} 1 & 3 & 2 \\ 4 & 2 & 5 \\ -2 & 1 & -2 \end{pmatrix} \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}}_{\begin{pmatrix} 3 & 10 & 4 \\ 12 & 10 & 10 \\ -6 & 1 & -4 \end{pmatrix}} \underbrace{\begin{pmatrix} 1 & 3 & 2 \\ 4 & 2 & 5 \\ -2 & 1 & -2 \end{pmatrix}^{-1}}_{\begin{pmatrix} -9 & 8 & 11 \\ -2 & 2 & 3 \\ 8 & -7 & -10 \end{pmatrix}}$$

$$= \begin{pmatrix} 3 & 10 & 4 \\ 12 & 10 & 10 \\ -6 & 1 & -4 \end{pmatrix} \begin{pmatrix} -9 & 8 & 11 \\ -2 & 2 & 3 \\ 8 & -7 & -10 \end{pmatrix}$$

$$= \begin{pmatrix} -15 & 16 & 23 \\ -48 & 46 & 62 \\ 20 & -18 & -23 \end{pmatrix}$$

2. *(30 pts)* Find a Jordan basis and the Jordan canonical form for the matrix

$$\begin{pmatrix} 7 & 6 & -7 \\ -5 & -1 & 6 \\ 1 & 3 & 0 \end{pmatrix}$$

$$\det(A-\lambda I) = \det\begin{pmatrix} 7-\lambda & 6 & -7 \\ -5 & -1-\lambda & 6 \\ 1 & 3 & -\lambda \end{pmatrix}$$

$$= 1(29-7\lambda) - 3(7-6\lambda) - \lambda(23-6\lambda+\lambda^2)$$

$$= 8 - 12\lambda + 6\lambda^2 - \lambda^3$$

$$= -(\lambda-2)^3$$ ← only 1 eigenvalue, $\lambda = 2$

To find eigenvectors, row reduce $A-2I = M$:

$$\left(\begin{array}{ccc|ccc} 5 & 6 & -7 & 1 & 0 & 0 \\ -5 & -3 & 6 & 0 & 1 & 0 \\ 1 & 3 & -2 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 3 & -2 & 0 & 0 & 1 \\ 5 & 6 & -7 & 1 & 0 & 0 \\ 0 & 3 & -1 & 1 & 1 & 0 \end{array}\right)\begin{array}{l} (3) \\ (1) \\ (2)+(1) \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 3 & -2 & 0 & 0 & 1 \\ 0 & -9 & 3 & 1 & 0 & -5 \\ 0 & 3 & -1 & 1 & 1 & 0 \end{array}\right)\begin{array}{l} (1) \\ (2)-5(1) \\ (3) \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 4 & 3 & -5 \\ 0 & 3 & -1 & 1 & 1 & 0 \end{array}\right)\begin{array}{l} (1)-(3) \\ (2)+3(3) \\ (3) \end{array}$$

$$\underbrace{\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1/3 \\ 0 & 0 & 0 \end{pmatrix}}_{R}\underbrace{\begin{pmatrix} -1 & -1 & 1 \\ 1/3 & 1/3 & 0 \\ 4 & 3 & -5 \end{pmatrix}}_{E}\begin{array}{l} (1) \\ (3)/3 \\ (2) \end{array}$$

$\vec{v}_1 = \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix}$ is the only eigenvector

The Jordan form must be

$$J = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

(over →)

(extra space if needed)

To find $\vec{v}_2$, we solve

$$M\vec{v}_2 = \vec{v}_1$$

$$R\vec{v}_2 = E\vec{v}_1$$

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1/3 \\ 0 & 0 & 0 \end{pmatrix}\vec{v}_2 = \begin{pmatrix} -1 & -1 & 1 \\ 1/3 & 1/3 & 0 \\ 4 & 3 & -5 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 4/3 \\ 0 \end{pmatrix}$$

Choosing $z = 0$, we get $\vec{v}_2 = \begin{pmatrix} -1 \\ 4/3 \\ 0 \end{pmatrix}$

To find $\vec{v}_3$, we solve

$$M\vec{v}_3 = \vec{v}_2$$

$$R\vec{v}_3 = E\vec{v}_2$$

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1/3 \\ 0 & 0 & 0 \end{pmatrix}\vec{v}_3 = \begin{pmatrix} -1 & -1 & 1 \\ 1/3 & 1/3 & 0 \\ 4 & 3 & -5 \end{pmatrix}\begin{pmatrix} -1 \\ 4/3 \\ 0 \end{pmatrix} = \begin{pmatrix} -1/3 \\ 1/9 \\ 0 \end{pmatrix}$$

Choosing $z=0$, we get $\vec{v}_3 = \begin{pmatrix} -1/3 \\ 1/9 \\ 0 \end{pmatrix}$

So $\left\{ \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 4/3 \\ 0 \end{pmatrix}, \begin{pmatrix} -1/3 \\ 1/9 \\ 0 \end{pmatrix} \right\}$ is a Jordan basis.

3. *(10 pts)* Show by direct calculation that the Hermitian transpose, defined by $A^* = \bar{A}^T$, satisfies the following equation.

$$\langle A\vec{v}, \vec{w}\rangle_H = \langle \vec{v}, A^*\vec{w}\rangle_H$$

$$\langle A\vec{v}, \vec{w}\rangle_H = (A\vec{v})^T \overline{\vec{w}}$$

$$= (\vec{v}^T A^T)(\overline{\vec{w}})$$

$$= \vec{v}^T (A^T \overline{\vec{w}})$$

$$= \vec{v}^T \left(\overline{\bar{A}^T \vec{w}} \right)$$

$$= \langle \vec{v}, \bar{A}^T \vec{w}\rangle_H$$

$$= \langle \vec{v}, A^* \vec{w}\rangle_H$$

4. *(20 pts)* Find a fundamental set of solutions to the system of differential equations described by $\vec{y}' = A\vec{y}$, where

$$A = \begin{pmatrix} 4 & 1 & -3 \\ 43 & -7 & 191 \\ 3 & -1 & 18 \end{pmatrix}$$

P

Here is some arithmetic that you might find useful:

$$A = \begin{pmatrix} 4 & 1 & -3 \\ 43 & -7 & 191 \\ 3 & -1 & 18 \end{pmatrix} = \begin{pmatrix} 5 & 2 & 1 \\ 2 & 7 & 3 \\ -1 & 0 & 0 \end{pmatrix} \overset{=J}{\begin{pmatrix} 5 & 1 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{pmatrix}} \overset{=P^{-1}}{\begin{pmatrix} 0 & 0 & -1 \\ -3 & 1 & -13 \\ 7 & -2 & 31 \end{pmatrix}}$$

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 5 & 2 & 1 \\ 2 & 7 & 3 \\ -1 & 0 & 0 \end{pmatrix}}_{P} \underbrace{\begin{pmatrix} 0 & 0 & -1 \\ -3 & 1 & -13 \\ 7 & -2 & 31 \end{pmatrix}}_{P^{-1}}$$

The above tells us that

$$J = P^{-1}AP$$

So J is the Jordan canonical form for A, and the columns of P,

$$\vec{v}_1 = \begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} 2 \\ 7 \\ 0 \end{pmatrix} \quad \vec{v}_3 = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$$

form a Jordan basis.

So a fundamental set of solutions is $\{e^{xA}\vec{v}_1, e^{xA}\vec{v}_2, e^{xA}\vec{v}_3\}$.

$$e^{xA}\vec{v}_1 = e^{5x}\vec{v}_1 = e^{5x}\begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix}$$

$$e^{xA}\vec{v}_2 = e^{5x}(\vec{v}_2 + x\vec{v}_1) = e^{5x}\begin{pmatrix} 2+5x \\ 7+2x \\ 0-1x \end{pmatrix}$$

$$e^{xA}\vec{v}_3 = e^{5x}\left(\vec{v}_3 + x\vec{v}_2 + \tfrac{1}{2}x^2\vec{v}_1\right) = e^{5x}\begin{pmatrix} 1+2x+\frac{5}{2}x^2 \\ 3+7x+x^2 \\ -\frac{1}{2}x^2 \end{pmatrix}$$

5. *(20 pts)* Find a particular solution to the system of differential equations $\vec{y}' = A\vec{y} + \vec{g}$, with

$$A = \begin{pmatrix} -5 & -3 \\ 2 & 0 \end{pmatrix} \quad \text{and} \quad \vec{g}(x) = \begin{bmatrix} 0 \\ x \end{bmatrix}$$

(In case it might be useful, note that the vectors $(3, -2)$ *and* $(-1, 1)$ *are eigenvectors.)*

We guess $\vec{y} = \vec{a} + \vec{b}x$. The equation becomes

$$(\vec{a} + \vec{b}x)' = A(\vec{a} + \vec{b}x) + x\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\vec{b} = A\vec{a} + \left(A\vec{b} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)x$$

So we must have

$$A\vec{b} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \quad \text{and} \quad A\vec{a} = \vec{b}$$

$$\Downarrow$$

$$\vec{b} = A^{-1}\begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$\vec{b} = \frac{\begin{pmatrix} 0 & 3 \\ -2 & -5 \end{pmatrix}}{6}\begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$\vec{b} = \begin{pmatrix} -1/2 \\ 5/6 \end{pmatrix}$$

$$\Downarrow$$

$$\vec{a} = A^{-1}\begin{pmatrix} -1/2 \\ 5/6 \end{pmatrix}$$

$$\vec{a} = \frac{\begin{pmatrix} 0 & 3 \\ -2 & -5 \end{pmatrix}}{6}\begin{pmatrix} -1/2 \\ 5/6 \end{pmatrix}$$

$$\vec{a} = \begin{pmatrix} 5/12 \\ -19/36 \end{pmatrix}$$

So our particular solution is

$$\vec{y}_p = \vec{a} + \vec{b}x = \begin{pmatrix} 5/12 - \frac{1}{2}x \\ -19/36 + 5/6\, x \end{pmatrix}$$