EXAM 2

Math 107, 2010-2011 Spring, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.
All answers must be simplified. All of the policies and guidelines
on the class webpages are in effect on this exam.

Good luck!

Name Solutions

ID number____________________

1. __________
2. __________
3. __________
4. __________
5. __________
6. __________

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ____________________

Total Score __________ (/100 points)

1. *(15 pts)* The matrix A and its reduced row echelon form R are given below. Find bases for the nullspace of A, the row space of A, and the column space of A.

$$A = \begin{pmatrix} 1 & 5 & 23 & 3 & 25 & 2 \\ 5 & -2 & 7 & 4 & 32 & 9 \\ 0 & 8 & 32 & 3 & 26 & 6 \\ 10 & 1 & 34 & 2 & 33 & 6 \end{pmatrix} \qquad R = \begin{pmatrix} 1 & 0 & 3 & 0 & 2 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} -3x_3 - 2x_5 \\ -4x_3 - x_5 \\ x_3 \\ -6x_5 \\ x_5 \\ 0 \end{pmatrix} = x_3 \begin{pmatrix} -3 \\ -4 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -2 \\ -1 \\ 0 \\ -6 \\ 1 \\ 0 \end{pmatrix}$$

So $\left\{ \begin{pmatrix} -3 \\ -4 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ -1 \\ 0 \\ -6 \\ 1 \\ 0 \end{pmatrix} \right\}$ is a basis for $NS(A)$

$\left\{ \begin{pmatrix} 1 \\ 0 \\ 3 \\ 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 4 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 6 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a basis for $RS(A)$

$\left\{ \begin{pmatrix} 1 \\ 5 \\ 0 \\ 10 \end{pmatrix}, \begin{pmatrix} 5 \\ -2 \\ 8 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 9 \\ 6 \\ 6 \end{pmatrix} \right\}$ is a basis for $CS(A)$

2. *(15 pts)* Suppose you deposit \$100 per month continuously into an account that earns 5% per year. You begin this process upon the birth of your first child, and stop on that child's 18th birthday to make the account a birthday present. What will be the final balance in the account?

$(\$100/\text{mo} = \$1200/\text{yr})$

$$\frac{dB}{dt} = rB + 1200$$

let $z = rB + 1200$

$z' = rB' = rz$

$$z' = rz$$

$$\Rightarrow z = Ae^{rt}$$

$$rB + 1200 = Ae^{rt}$$

$$B = \frac{Ae^{rt} - 1200}{r}$$

$B(0) = 0$

$$\Rightarrow 0 = \frac{A - 1200}{r}$$

$$\Rightarrow A = 1200$$

$$B = \frac{1200}{r}(e^{rt} - 1)$$

$$B = 24000\left(e^{(.05)t} - 1\right)$$

$$B(18) = 24{,}000\left(e^{(.9)} - 1\right)$$

3. *(20 pts)* Suppose that $\vec{v}_1, \ldots, \vec{v}_n$ are in the vector space V, and $T : V \to W$ is a linear transformation. Show that if $\{T(\vec{v}_1), \ldots, T(\vec{v}_n)\}$ is linearly independent in W, then $\{\vec{v}_1, \ldots, \vec{v}_n\}$ must be linearly independent in V.

Suppose that $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is linearly dependent.

Then for some $c_1, \ldots, c_n$, not all zero, we have

$$c_1\vec{v}_1 + \cdots + c_n\vec{v}_n = \vec{0}$$

Applying T, and using linearity,

$$T(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n) = \vec{0}$$

$$c_1 T(\vec{v}_1) + \cdots + c_n T(\vec{v}_n) = \vec{0}$$

This would imply $\{T(\vec{v}_1), \ldots, T(\vec{v}_n)\}$ is dependent, contradicting the given statement.

So our assumption that $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is dependent must be wrong, and thus $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is independent. ∎

4. *(20 pts)* Find a real fundamental set of solutions to the fourth order differential equation

$$y'''' + 2y'' - 8y' + 5y = 0$$

$p(\lambda) = \lambda^4 + 2\lambda^2 - 8\lambda + 5$

$p(1) = 0 \implies (\lambda - 1)$ is a factor

$$\begin{array}{r|l} & \lambda^3 + \lambda^2 + 3\lambda - 5 \\ \lambda - 1 & \lambda^4 + 0\lambda^3 + 2\lambda^2 - 8\lambda + 5 \\ & \underline{\lambda^4 - \lambda^3} \\ & \lambda^3 + 2\lambda^2 - 8\lambda + 5 \\ & \underline{\lambda^3 - \lambda^2} \\ & 3\lambda^2 - 8\lambda + 5 \\ & \underline{3\lambda^2 - 3\lambda} \\ & -5\lambda + 5 \\ & \underline{-5\lambda + 5} \\ & 0 \end{array}$$

$g_1(\lambda) = \lambda^3 + \lambda^2 + 3\lambda - 5$

$g_1(1) = 0 \implies (\lambda - 1)$ is a factor

$$\begin{array}{r|l} & \lambda^2 + 2\lambda + 5 \\ \lambda - 1 & \lambda^3 + \lambda^2 + 3\lambda - 5 \\ & \underline{\lambda^3 - \lambda^2} \\ & 2\lambda^2 + 3\lambda - 5 \\ & \underline{2\lambda^2 - 2\lambda} \\ & 5\lambda - 5 \\ & \underline{5\lambda - 5} \\ & 0 \end{array}$$

$g_2(\lambda) = \lambda^2 + 2\lambda + 5$

roots are: $\dfrac{-2 \pm \sqrt{4 - 20}}{2}$

$= -1 \pm 2i$

So $p(\lambda) = (\lambda - 1)^2 (\lambda - (-1 + 2i))(\lambda - (-1 - 2i))$

Fund. set of solutions is $\{ e^x, xe^x, e^{-x}\cos 2x, e^{-x}\sin 2x \}$

5. *(15 pts)* Find a particular solution to the differential equation

$$y'' - 7y' + 12y = e^{3x}$$

$p(\lambda) = \lambda^2 - 7\lambda + 12 = (\lambda-3)(\lambda-4)$

3 is a root of multiplicity 1.

So our guess is $y_p = Axe^{3x}$

$y_p' = Ae^{3x} + Ax(3e^{3x})$

$y_p'' = 3Ae^{3x} + 3A(e^{3x} + 3xe^{3x})$

$= 6Ae^{3x} + 9Axe^{3x}$

The DE becomes

$$(6Ae^{3x} + 9Axe^{3x}) - 7(Ae^{3x} + 3Axe^{3x}) + 12(Axe^{3x}) = e^{3x}$$

$$\underbrace{(6A - 7A)}_{=1}e^{3x} + \underbrace{(9A - 21A + 12A)}_{=0 \checkmark}xe^{3x} = e^{3x}$$

$-A = 1 \implies A = -1$

So the particular solution is

$$y_p = -xe^{3x}$$

6. *(15 pts)* Bob likes to sing in the shower, but he is frustrated by the gain he experiences at the shower's natural frequency of ω_0. He is considering purchasing sound deadening material for the ceiling of the shower, with an "absorbance factor" (at his natural frequency) of k/ω_0, which will change the differential equation describing the vibrations to

$$u'' + \frac{k}{\omega_0}u' + \omega_0^2 u = a\cos\omega_0 t$$

where the function $a\cos\omega_0 t$ represents his voice when he sings at the natural frequency.

If Bob wants the gain at the natural frequency to be at most 1.2, what minimum value of k does he require?

Consider $z'' + \frac{k}{w_0} z' + w_0^2 z = a e^{iw_0 t}$

with $u = \mathrm{Re}(z)$.

Guess $z = Tae^{iw_0 t}$. Then the DE becomes

$$-w_0^2 Tae^{iw_0 t} + \frac{k}{w_0} w_0 i Tae^{iw_0 t} + w_0^2 Tae^{iw_0 t} = ae^{iw_0 t}$$

$$kiT ae^{iw_0 t} = ae^{iw_0 t}$$

$$T = \frac{1}{ki} = \left(\frac{1}{k}\right) e^{-i\pi/2}$$

$$\Rightarrow z = \left(\frac{1}{k}\right) a e^{i(w_0 t - \pi/2)}$$

$$\Rightarrow u = \left(\frac{1}{k}\right) a\cos\left(w_0 t - \frac{\pi}{2}\right)$$

$$\text{Gain} = \left(\frac{1}{k}\right) = 1.2$$

$$\Rightarrow k = \frac{1}{1.2} = \boxed{\frac{5}{6}}$$