EXAM 1

Math 107, 2010-2011 Spring, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.
All answers must be simplified. All of the policies and guidelines
on the class webpages are in effect on this exam.

Good luck!

Name Solutions

ID number ____________

1. ____________
2. ____________
3. ____________
4. ____________
5. ____________
6. ____________
7. ____________

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ____________

Total Score ____________ (/100 points)

1. *(15 pts)* Find the complete set of solutions to the system of equations below.

$$\underbrace{\begin{aligned} x_1 + 2x_2 - x_3 + 2x_4 \\ 2x_1 + 4x_2 - x_3 + 6x_4 - 2x_5 \\ 11x_1 + 22x_2 - 7x_3 + 30x_4 - 11x_5 \\ -3x_1 - 6x_2 + 2x_3 - 8x_4 + 3x_5 \end{aligned}}_{A\vec{x}} \begin{aligned} &= \\ &= \\ &= \\ &= \end{aligned} \underbrace{\begin{aligned} 13 \\ 19 \\ 100 \\ -27 \end{aligned}}_{\vec{b}}$$

You may use the arithmetic fact below, and also that the determinant of the left matrix on the left side of the equation below is -1.

$$\underbrace{\begin{pmatrix} 1 & 3 & 0 & 2 \\ 0 & 2 & 1 & 5 \\ 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 3 \end{pmatrix}}_{E} \underbrace{\begin{pmatrix} 1 & 2 & -1 & 2 & 0 \\ 2 & 4 & -1 & 6 & -2 \\ 11 & 22 & -7 & 30 & -11 \\ -3 & -6 & 2 & -8 & 3 \end{pmatrix}}_{A} = \underbrace{\begin{pmatrix} 1 & 2 & 0 & 4 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}}_{R}$$

det $\neq 0$
$\Rightarrow$ invtbl
$\Rightarrow$ prod. of elem. mats.

$$A\vec{x} = \vec{b} \iff E A \vec{x} = E\vec{b} \iff R\vec{x} = E\vec{b}$$

(since E invertible)

So we can equivalently solve

$$\begin{pmatrix} 1 & 2 & 0 & 4 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 16 \\ 3 \\ 5 \\ 0 \end{pmatrix}$$

$$\begin{aligned} x_1 &= 16 - 2x_2 - 4x_4 \\ x_3 &= 3 \quad\quad\quad\; - 2x_4 \\ x_5 &= 5 \end{aligned}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 16 - 2x_2 - 4x_4 \\ x_2 \\ 3 - 2x_4 \\ x_4 \\ 5 \end{pmatrix} = \boxed{\begin{pmatrix} 16 \\ 0 \\ 3 \\ 0 \\ 5 \end{pmatrix} + x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -4 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix}}$$

2. *(10 pts)* The rows of the 5x5 nonsingular matrices A and B have the following in common:

(a) B_1 is A_1 plus 2 times A_2.

(b) B_2 is 5 times A_5 minus 6 times A_3.

(c) B_3 is 8 times A_2 minus A_1.

(d) B_4 is 3 times A_5 minus A_4.

(e) B_5 is A_4.

Find the unique matrix C that satisfies $CA = B$.

$$\begin{pmatrix} & & \\ & C & \\ & & \end{pmatrix}\begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{pmatrix} = \begin{pmatrix} A_1 + 2A_2 \\ -6A_3 + 5A_5 \\ -A_1 + 8A_2 \\ -A_4 + 3A_5 \\ A_4 \end{pmatrix}$$

The rows of A are independent because A is nonsingular; so the expressions using them to describe the rows of B are unique.

We can find the rows of C by how each corresponding row of B is written as a linear combination of the rows of A.

So:

$$\boxed{C = \begin{pmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & -6 & 0 & 5 \\ -1 & 8 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}}$$

3. *(15 pts)* Find the inverse matrix for the matrix N below.

$$N = \begin{pmatrix} 5 & 2 & 2 \\ 1 & -5 & 0 \\ -3 & 2 & -1 \end{pmatrix}$$

We row reduce $(N|I)$:

$$\left(\begin{array}{ccc|ccc} 5 & 2 & 2 & 1 & 0 & 0 \\ 1 & -5 & 0 & 0 & 1 & 0 \\ -3 & 2 & -1 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & -5 & 0 & 0 & 1 & 0 \\ 5 & 2 & 2 & 1 & 0 & 0 \\ -3 & 2 & -1 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} 2 \\ 1 \\ 3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & -5 & 0 & 0 & 1 & 0 \\ 0 & 27 & 2 & 1 & -5 & 0 \\ 0 & -13 & -1 & 0 & 3 & 1 \end{array}\right) \begin{array}{l} 1 \\ 2 - 5 \cdot 1 \\ 3 + 3 \cdot 1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & -5 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 2 \\ 0 & -13 & -1 & 0 & 3 & 1 \end{array}\right) \begin{array}{l} 1 \\ 2 + 2 \cdot 3 \\ 3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 5 & 6 & 10 \\ 0 & 1 & 0 & 1 & 1 & 2 \\ 0 & 0 & -1 & 13 & 16 & 27 \end{array}\right) \begin{array}{l} 1 + 5 \cdot 2 \\ 2 \\ 3 + 13 \cdot 2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 5 & 6 & 10 \\ 0 & 1 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & -13 & -16 & -27 \end{array}\right) \begin{array}{l} 1 \\ 2 \\ -3 \end{array}$$

this is I, so N is invertible

$$N^{-1} = \begin{pmatrix} 5 & 6 & 10 \\ 1 & 1 & 2 \\ -13 & -16 & -27 \end{pmatrix}$$

4. *(15 pts)* Show that the square matrix A is nonsingular ($\text{rref}(A) = I$) if and only if A is invertible.

($\Rightarrow$) We assume A is nonsingular, so $\text{rref}(A) = I$. The row reduction is written with elementary matrices as:

$$E_k \cdots E_1 A = R = I$$

Writing $E_k \cdots E_1 = E$, we have

$$EA = I$$

So A is invertible and $A^{-1} = E$.

($\Leftarrow$) We assume A is invertible.
To show A is nonsingular, it is enough to show $\text{rref}(A)$ has a pivot in every row; or equivalently that $A\vec{x} = \vec{b}$ has universal existence.

We confirm this by observing that $\vec{x} = A^{-1}\vec{b}$ is a solution:

$$A(A^{-1}\vec{b}) = (AA^{-1})\vec{b} = \vec{b} \checkmark$$

So A is nonsingular. ∎

5. *(15 pts)* Use permutations to compute the determinant of the matrix

$$A = \begin{pmatrix} 5 & -6 & 10 \\ 1 & 3 & -2 \\ 2 & 4 & 1 \end{pmatrix}$$

$$\det A = \sum_{\alpha \in S_3} \operatorname{sgn}(\alpha)\, a_{1\alpha(1)}\, a_{2\alpha(2)}\, a_{3\alpha(3)}$$

We can enumerate the six permutations in S_3 and compute these terms individually:

$1\to1,\ 2\to2,\ 3\to3$	sgn $= +1$	$\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$	$(+)(5)(3)(1)$
$1\to2,\ 2\to1,\ 3\to3$	sgn $= -1$	$\begin{pmatrix} & 1 & \\ 1 & & \\ & & 1 \end{pmatrix}$	$(-)(-6)(1)(1)$
$1\to3,\ 2\to2,\ 3\to1$	sgn $= -1$	$\begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}$	$(-)(10)(3)(2)$
$1\to1,\ 2\to3,\ 3\to2$	sgn $= -1$	$\begin{pmatrix} 1 & & \\ & & 1 \\ & 1 & \end{pmatrix}$	$(-)(5)(-2)(4)$
$1\to2,\ 2\to3,\ 3\to1$	sgn $= +1$	$\begin{pmatrix} & 1 & \\ & & 1 \\ 1 & & \end{pmatrix}$	$(+)(-6)(-2)(2)$
$1\to3,\ 2\to1,\ 3\to2$	sgn $= +1$	$\begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \end{pmatrix}$	$(+)(10)(1)(4)$

$$65$$

So $\boxed{\det A = 65}$

6. *(15 pts)* We consider here the matrix R below. Compute the entry in the 3rd row and 4th column of the matrix R^{-1}.

$$R = \begin{pmatrix} 2 & 2 & 2 & 3 \\ 1 & 1 & 1 & 1 \\ 1 & 3 & 2 & 6 \\ 4 & -2 & -4 & 1 \end{pmatrix}$$

$$R^{-1} = \frac{\operatorname{adj}(R)}{\det(R)} = \frac{C^T}{\det(R)}$$

So the 3rd row, 4th column entry of R^{-1} is $\dfrac{C_{43}}{\det(R)}$

$$C_{43} = (-1)^{4+3} \det\begin{pmatrix} 2 & 2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & 6 \end{pmatrix} = (-1)\big((2)(3) - (2)(5) + (3)(2)\big)$$
$$= -2$$

To compute $\det R$, we first subtract 2 times the second row from the first row (does not change det!) to get

$$R' = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 3 & 2 & 6 \\ 4 & -2 & -4 & 1 \end{pmatrix}$$

$$\det R = \det R' = (-1)\det\begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 2 \\ 4 & -2 & -4 \end{pmatrix} = (-1)\big((1)(-8) - (1)(-12) + (1)(-14)\big)$$
$$= 10$$

So the entry in question is

$$\frac{C_{43}}{\det(R)} = \frac{-2}{10} = \boxed{\frac{-1}{5}}$$

7. *(15 pts)* You and your friend Bob are considering the collection $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ of vectors in $\mathbb{R}^5$, where

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \\ 4 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 2 \\ 4 \\ 5 \\ 7 \end{bmatrix} \quad \vec{v}_3 = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 5 \\ 12 \end{bmatrix} \quad \vec{v}_4 = \begin{bmatrix} 3 \\ -4 \\ 0 \\ 1 \\ 10 \end{bmatrix}$$

Bob observes, correctly, that none of these four vectors is a nontrivial linear combination of the other vectors. He cites this fact *alone* as demonstrating that this collection of vectors is linearly independent.

Is Bob's reasoning correct? Explain why or why not. Is Bob's conclusion correct? Explain why or why not.

Bob's reasoning is not correct. His observation does not eliminate the possibility that one of the vectors might be a trivial linear combination of the other vectors — which, if it were to be the case, would make the vectors dependent.

Bob's conclusion is correct though. We can eliminate the possibility of a trivial linear combination by noting that this would mean the vector would be the zero vector, which is clearly not the case.

This, combined with Bob's observation, shows that none of these vectors is any sort of linear combination of the other vectors; so, the collection is linearly independent.