EXAM 3

Math 107, 2010-2011 Fall, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.
All answers must be simplified. All of the policies and guidelines
on the class webpages are in effect on this exam.

Good luck!

Name Solutions

ID number____________________

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ____________________

1. __________

2. __________

3. __________

4. __________

5. __________

Total Score __________ (/100 points)

1. *(15 pts)* Find a particular solution to the system below.

$$\begin{aligned} y_1' &= 3y_1 - 2y_2 + 4x \\ y_2' &= 2y_1 - y_2 + 3 \end{aligned}$$

In vector form this equation is

$$\vec{y}\,' = A\vec{y} + \vec{v}_1 + x\vec{v}_2$$

with $A = \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}$, $\vec{v}_1 = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$

We guess the form of the particular solution as:

$$\vec{y} = \vec{a} + x\vec{b}$$

Then $\vec{y}\,' = \vec{b}$

$$A\vec{y} = A\vec{a} + xA\vec{b}$$

The equation becomes

$$\vec{b} = A\vec{a} + xAb + \vec{v}_1 + x\vec{v}_2$$

So $A\vec{a} = \vec{b} - \vec{v}_1$ and $A\vec{b} = -\vec{v}_2$

Then $\vec{b} = -A^{-1}\vec{v}_2 = -\dfrac{\begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}}{1}\begin{pmatrix} 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \end{pmatrix}$

$$\vec{a} = A^{-1}(\vec{b} - \vec{v}_1) = \frac{\begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}}{1}\begin{pmatrix} 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 6 \\ 7 \end{pmatrix}$$

So we have $\vec{y}_p = \begin{pmatrix} 6 + 4x \\ 7 + 8x \end{pmatrix}$

2. *(20 pts)* We consider here the inner product space $C^0[0,2]$ using the L^2 inner product. Let V be the subspace spanned by the functions $f_1 = 3$, $f_2 = x^2$. Use the Gram-Schmidt process to find an orthonormal basis for V.

$$\langle f, g\rangle = \int_0^2 f g \, dx \qquad \|f\| = \sqrt{\int_0^2 f^2 \, dx}$$

$\underline{\vec{v}_1:}$

$$\vec{v}_1 = \frac{f_1}{\|f_1\|} \qquad \|f_1\| = \sqrt{\int_0^2 3^2 \, dx} = 3\sqrt{2}$$

$$\vec{v}_1 = \frac{3}{3\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$\underline{\vec{v}_2:}$

$$\vec{x}_2 = f_2 - \langle f_2, \vec{v}_1\rangle \vec{v}_1 \qquad \langle f_2, \vec{v}_1\rangle = \int_0^2 (x^2)\left(\tfrac{1}{\sqrt{2}}\right) dx = \frac{8}{3\sqrt{2}}$$

$$\vec{x}_2 = x^2 - \left(\frac{8}{3\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) = x^2 - \frac{4}{3}$$

$$\|\vec{x}_2\|^2 = \int_0^2 \left(x^2 - \tfrac{4}{3}\right)^2 dx = \int_0^2 x^4 - \tfrac{8}{3}x^2 + \tfrac{16}{9} \, dx$$

$$= \left(\tfrac{1}{5}x^5 - \tfrac{8}{9}x^3 + \tfrac{16}{9}x\right]_0^2 = \frac{32}{5} - \frac{64}{9} + \frac{32}{9}$$

$$= \frac{128}{45}$$

$$\|\vec{x}_2\| = \sqrt{\frac{128}{45}} = \frac{8}{3}\sqrt{\frac{2}{5}}$$

$$\vec{v}_2 = \frac{\vec{x}_2}{\|\vec{x}_2\|} = \frac{x^2 - \frac{4}{3}}{\frac{8}{3}\sqrt{\frac{2}{5}}} = \frac{3}{8}\sqrt{\frac{5}{2}}\, x^2 - \frac{1}{2}\sqrt{\frac{5}{2}}$$

So $\left\{ \frac{1}{\sqrt{2}}, \ \frac{3}{8}\sqrt{\frac{5}{2}}\, x^2 - \frac{1}{2}\sqrt{\frac{5}{2}} \right\}$ is an orthonormal basis for V.

3. *(15 pts)* The matrix A has eigenvalues 5, 2, and 7, and corresponding eigenvectors $(1, 2, 1)$, $(3, -1, -2)$, $(2, 1, 0)$. Find a fundamental set of solutions to the system of equations

$$\vec{y}' = A\vec{y}$$

and also find the solution satisfying the initial condition $\vec{y}(0) = (3, 2, 5)$.

$$\lambda_1 = 5 \qquad \vec{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

$$\lambda_2 = 2 \qquad \vec{v}_2 = \begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix}$$

$$\lambda_3 = 7 \qquad \vec{v}_3 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

A fundamental set of solutions is $\{e^{xA}\vec{v}_1, e^{xA}\vec{v}_2, e^{xA}\vec{v}_3\}$,

which we evaluate by

$$\left\{ e^{5x}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, e^{2x}\begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix}, e^{7x}\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \right\} = \{f_1, f_2, f_3\}$$

At $x=0$, we have $f_i(0) = \vec{v}_i$, so we need to solve

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 & 2 \\ 2 & -1 & 1 \\ 1 & -2 & 0 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix}$$

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} -2 & 4 & -5 \\ -1 & 2 & -3 \\ 3 & -5 & 7 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} -23 \\ -14 \\ 34 \end{pmatrix}$$

So our I.V.P. solution is

$$\vec{y} = c_1 f_1 + c_2 f_2 + c_3 f_3$$

$$= -23e^{5x}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} - 14e^{2x}\begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix} + 34e^{7x}\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

4. *(30 pts)* We consider here the system of equations $\vec{y}' = A\vec{y}$, with

$$A = \begin{pmatrix} 5 & -1 & -4 \\ 2 & 2 & -4 \\ 2 & -1 & 0 \end{pmatrix}$$

The eigenvalues are 3 and 2, and the characteristic polynomial is $p(\lambda) = \lambda^3 - 7\lambda^2 + 16\lambda - 12$. Find a fundamental set of solutions to this system of equations.

$$\begin{array}{r|l} & \lambda^2 - 4\lambda + 4 \\ \lambda - 3 & \lambda^3 - 7\lambda^2 + 16\lambda - 12 \\ & \lambda^3 - 3\lambda^2 \\ \hline & -4\lambda^2 + 16\lambda - 12 \\ & -4\lambda^2 + 12\lambda \\ \hline & 4\lambda - 12 \\ & 4\lambda - 12 \\ \hline & 0 \end{array}$$

And $\lambda^2 - 4\lambda + 4 = (\lambda-2)^2$

So

$$p(\lambda) = (\lambda-3)(\lambda-2)^2$$

To find the eigenvector for $\lambda = 3$, we solve

$$(A - 3I)\vec{v}_1 = \vec{0} \qquad \begin{pmatrix} 2 & -1 & -4 \\ 2 & -1 & -4 \\ 2 & -1 & -3 \end{pmatrix}\vec{v}_1 = \vec{0}$$

The nullspace has dim $= 1$, and by inspection $\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$.

For $\lambda = 2$, we solve

$$(A - 2I)\vec{v}_2 = \vec{0} \qquad \begin{pmatrix} 3 & -1 & -4 \\ 2 & 0 & -4 \\ 2 & -1 & -2 \end{pmatrix}\vec{v}_2 = \vec{0}$$

$$\left(\begin{array}{ccc|c} 3 & -1 & -4 & 0 \\ 2 & 0 & -4 & 0 \\ 2 & -1 & -2 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 1 & 0 & -2 & 0 \\ 0 & -1 & 2 & 0 \end{array}\right) \begin{array}{l} ① - ② \\ ②/2 \\ ③ - ② \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} ① \\ ③ \\ ② - ① + ③ \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} ① - ② \\ -② \\ ③ \end{array}$$

rank $= 2$ so there is only 1 evect,

$$\vec{v}_2 = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$$

The Jordan form then must be

$$\begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

To find the third Jordan basis vector, we solve

$$(A-2I)\vec{v}_3 = \vec{v}_2 \qquad \begin{pmatrix} 3 & -1 & -4 \\ 2 & 0 & -4 \\ 2 & -1 & -2 \end{pmatrix}\vec{v}_3 = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$$

By inspection, a convenient solution is $\vec{v}_3 = \begin{pmatrix} 0 \\ 0 \\ -1/2 \end{pmatrix}$

Our Jordan basis $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ then gives a fundamental set $\{e^{xA}\vec{v}_1, e^{xA}\vec{v}_2, e^{xA}\vec{v}_3\}$, which can be evaluated by

$$e^{xA}\vec{v}_1 = e^{3x}\vec{v}_1 = e^{3x}\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

$$e^{xA}\vec{v}_2 = e^{2x}\vec{v}_2 = e^{2x}\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$$

$$e^{xA}\vec{v}_3 = e^{2x}(\vec{v}_3 + x\vec{v}_2) = e^{2x}\begin{pmatrix} 2x \\ 2x \\ -\frac{1}{2} + x \end{pmatrix}$$

5. *(20 pts)* The 3×3 matrix A has eigenvectors

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix} \quad \vec{v}_1 = \begin{bmatrix} 6 \\ 2 \\ -3 \end{bmatrix} \quad \vec{v}_1 = \begin{bmatrix} 3 \\ -6 \\ 2 \end{bmatrix}$$

Show that A must be symmetric. (Make sure to explain all of your reasoning clearly.)

First we observe that $\vec{v}_1 \cdot \vec{v}_2 = 0$, $\vec{v}_1 \cdot \vec{v}_3 = 0$, $\vec{v}_2 \cdot \vec{v}_3 = 0$.

So these eigenvectors form an orthogonal basis, and normalizing by $\vec{w}_i = \frac{\vec{v}_i}{\|\vec{v}_i\|} = \frac{\vec{v}_i}{7}$, we have an orthonormal basis of eigen vectors.

Then A is diagonalizable, by

$D = P^{-1}AP$, where $P = \begin{pmatrix} | & | & | \\ \vec{w}_1 & \vec{w}_2 & \vec{w}_3 \\ | & | & | \end{pmatrix}$

So

$$A = PDP^{-1}$$

Since P is orthogonal we have $P^{-1} = P^T$, so

$$A = PDP^T$$

Then $A^T = (P^T)^T D^T P^T = PDP^T = A$

So A is symmetric.