EXAM 2

Math 107, 2010-2011 Fall, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.
All answers must be simplified. All of the policies and guidelines
on the class webpages are in effect on this exam.

Good luck!

Name Solutions

ID number____________________

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ____________________

1. __________

2. __________

3. __________

4. __________

5. __________

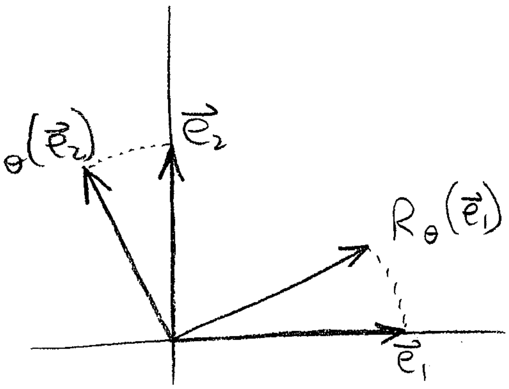
6. __________

7. __________

Total Score __________ (/100 points)

1. *(15 pts)* In class we showed that rotations around the origin in $\mathbb{R}^2$ are linear transformations. We also showed that the matrix for a composition of linear transformations is the product of the corresponding matrices. Use these facts to prove the angle addition formulas for the sin and cos functions.

$$[R_\theta] = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$



$$[R_a][R_b] = [R_a R_b] = [R_{a+b}]$$

$$\begin{pmatrix} \cos a & -\sin a \\ \sin a & \cos a \end{pmatrix}\begin{pmatrix} \cos b & -\sin b \\ \sin b & \cos b \end{pmatrix} = \begin{pmatrix} \cos(a+b) & -\sin(a+b) \\ \sin(a+b) & \cos(a+b) \end{pmatrix}$$

Multiplying out the matrix product on the left, we get

$$\cos a \cos b - \sin a \sin b = \cos(a+b)$$
$$\sin a \cos b + \cos a \sin b = \sin(a+b)$$

as desired.

2. *(15 pts)* Find a fundamental set of real solutions to the differential equation below.

$$y'''' - 8y''' + 25y'' - 36y' + 20y = 0$$

(Hint: You might make use of the fact that $y = e^{2x}\sin(x)$ is a solution.)

The hint tells us that $r = 2+i$ is a root of the char. poly.; then we know $\bar{r} = 2-i$ is also.

So $(\lambda-(2+i))(\lambda-(2-i)) = \lambda^2-4\lambda+5$ is a factor of the char. poly. Dividing, we get

$$\begin{array}{r|l} & \lambda^2 - 4\lambda + 4 \\ \hline \lambda^2-4\lambda+5 & \lambda^4 - 8\lambda^3 + 25\lambda^2 - 36\lambda + 20 \\ & \underline{\lambda^4 - 4\lambda^3 + 5\lambda^2} \\ & -4\lambda^3 + 20\lambda^2 - 36\lambda + 20 \\ & \underline{-4\lambda^3 + 16\lambda^2 - 20\lambda} \\ & 4\lambda^2 - 16\lambda + 20 \\ & \underline{4\lambda^2 - 16\lambda + 20} \\ & 0 \end{array}$$

Then $p(\lambda)$ factors as

$$p(\lambda) = (\lambda^2-4\lambda+5)(\lambda^2-4\lambda+4)$$
$$= (\lambda-(2+i))(\lambda-(2-i))(\lambda-2)^2$$

Our fundamental set of real solutions then is

$$\{e^{2x}\cos x,\ e^{2x}\sin x,\ e^{2x},\ xe^{2x}\}$$

3. *(15 pts)* Show that the functions x, e^x, $\sin x$, and $\cos x$ form a linearly independent set of functions in C^0.

$$W(x) = \det\begin{pmatrix} x & e^x & \sin x & \cos x \\ 1 & e^x & \cos x & -\sin x \\ 0 & e^x & -\sin x & -\cos x \\ 0 & e^x & -\cos x & \sin x \end{pmatrix}$$

$$W(0) = \det\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{pmatrix} = -(1)\det\begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}$$

$$= -\left((1)\det\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} - (0) + (1)\det\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}\right) = 2 \neq 0$$

So this is an independent set of fns.

4. *(15 pts)* A mass is intended to be placed on a spring in a resistive medium. The mass will be moved from the natural position and released, and then no external forces will act on the system.

The spring to be used exerts a force of 10N when stretched 0.25m from its natural length, and the resistance in the medium is 25N when the mass is moving at a speed of 0.5m/s. (Recall that $1N = 1kgm/s^2$.)

What is the largest mass that can be used in this system without having oscillations in the resulting motion?

(We use M for mass to avoid confusion with m for meters.)

$$My'' + fy' + ky = 0$$

roots of char. poly are $r = \dfrac{-f \pm \sqrt{f^2 - 4Mk}}{2M}$

no oscillation $\Rightarrow$ real roots $\Rightarrow f^2 - 4Mk > 0$

So largest mass is

$$M = \frac{f^2}{4k} = \frac{\left(25\,N / .5\left(\frac{m}{s}\right)\right)^2}{4\left(10\,N / .25\,m\right)} = \left(\frac{2500}{160}\right)\left(\frac{N^2/\left(\frac{m^2}{s^2}\right)}{N/m}\right)$$

$$= \frac{125}{8}\,\frac{N}{m/s^2} = \boxed{\frac{125}{8}\,kg}$$

5. *(15 pts)* Find a particular solution to the differential equation below.

$$y''' + 3y'' + 3y' + y = e^{-x}$$

(Hint: Factor the characteristic polynomial by recalling Pascal's triangle.)

$$p(\lambda) = \lambda^3 + 3\lambda^2 + 3\lambda + 1 = (\lambda+1)^3$$

For $g(x) = e^{-x}$, we have $r = a+bi = -1$

This is a root of $p(\lambda)$, with multiplicity 3.

So we guess $y = kx^3e^{-x}$

$$y' = k(3x^2 - x^3)e^{-x}$$

$$y'' = k\left((6x - 3x^2) - (3x^2 - x^3)\right)e^{-x}$$

$$= k(6x - 6x^2 + x^3)e^{-x}$$

$$y''' = k\left((6 - 12x + 3x^2) - (6x - 6x^2 + x^3)\right)e^{-x}$$

$$= k(6 - 18x + 9x^2 - x^3)e^{-x}$$

Then

$$L(y) = ke^{-x}\Big[(6 - 18x + 9x^2 - x^3) + 3(6x - 6x^2 + x^3) + 3(3x^2 - x^3) + (x^3)\Big]$$

$$= ke^{-x}[6] = e^{-x} \implies k = 1/6$$

So $y_p = \frac{1}{6}x^3e^{-x}$ is a particular solution

6. *(20 pts)* The linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is represented in the standard basis by the matrix $A = [T]_{\mathcal{S}}^{\mathcal{S}}$. We know also that

$$A \begin{bmatrix} 7 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix} \quad \text{and} \quad A \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$$

Use the matrix for T with respect to convenient bases to compute the matrix A.

Choose $\mathcal{V} = \left\{ \underbrace{\begin{pmatrix} 7 \\ 4 \end{pmatrix}}_{\vec{v}_1}, \underbrace{\begin{pmatrix} 2 \\ 1 \end{pmatrix}}_{\vec{v}_2} \right\}$, $\mathcal{W} = \left\{ \underbrace{\begin{pmatrix} 6 \\ 2 \end{pmatrix}}_{\vec{w}_1}, \underbrace{\begin{pmatrix} -3 \\ 5 \end{pmatrix}}_{\vec{w}_2} \right\}$

Then $[T]_{\mathcal{V}}^{\mathcal{W}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

And $[I]_{\mathcal{V}}^{\mathcal{S}} = \begin{pmatrix} 7 & 2 \\ 4 & 1 \end{pmatrix}$, $[I]_{\mathcal{W}}^{\mathcal{S}} = \begin{pmatrix} 6 & -3 \\ 2 & 5 \end{pmatrix}$

$$\text{So } A = [T]_{\mathcal{S}}^{\mathcal{S}} = [I]_{\mathcal{W}}^{\mathcal{S}} [T]_{\mathcal{V}}^{\mathcal{W}} [I]_{\mathcal{S}}^{\mathcal{V}}$$

$$= \begin{pmatrix} 6 & -3 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 7 & 2 \\ 4 & 1 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 6 & -3 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 4 & -7 \end{pmatrix}$$

$$= \begin{pmatrix} -18 & 33 \\ 18 & -31 \end{pmatrix}$$