EXAM 1

Math 107, 2010-2011 Fall, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.
All answers must be simplified. All of the policies and guidelines
on the class webpages are in effect on this exam.

Good luck!

Name Solutions ______________________

ID number______________________

1. ____________
2. ____________
3. ____________
4. ____________
5. ____________
6. ____________
7. ____________

"I have adhered to the Duke Community
Standard in completing this
examination."

Signature: ______________________

Total Score ____________ (/100 points)

This is a "log confidence" problem, as discussed in class and on the course webpage. In the box next to each response, indicate your confidence that this is the correct response. As discussed on the course webpage, all indicated confidences must be positive integers between 1 and 97, and the sum must be 100.

The scoring function is given by $f(x) = 3\log(x/25)$, where x is the confidence indicated for the correct response. Scores will be rounded to the nearest tenth of a point. A table of sample confidences and scores is given. You do NOT need to justify your answers on this problem.

1. Which ONE of the following is NOT a vector space? (For each, the proposed vector addition operation is indicated by the symbol "$\oplus$" and the proposed scalar multiplication is indicated by the symbol "$\odot$".)

	A. The set of C^1 functions $f : \mathbb{R}^1 \to \mathbb{R}^1$ with $f(5) = 0$ and $f'(2) = 0$, with operations defined by $(f \oplus g)(x) = f(x) + g(x)$ and $(c \odot f)(x) = cf(x)$
	B. The set of functions $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ representing rotations around the origin by an angle $\theta \in \mathbb{R}$, with operations defined by $R_{\theta_1} \oplus R_{\theta_2} = R_{\theta_1+\theta_2}$ and $c \odot R_\theta = R_{c\theta}$
✓	C. The set of real 2×2 matrices, with operations defined by $M_1 \oplus M_2 = M_1 M_2$ and $c \odot M = cM$
	D. The Euclidean space $\mathbb{R}^2$, with the usual operations, defined by $(x_1, y_1) \oplus (x_2, y_2) = (x_1+x_2, y_1+y_2)$ and $c \odot (x, y) = (cx, cy)$

Conf.	Score
1	-9.7
5	-4.8
10	-2.7
15	-1.5
20	-0.7
25	0
30	0.5
35	1
40	1.4
45	1.8
50	2.1
55	2.4
60	2.6
65	2.9
70	3.1
75	3.3
80	3.5
85	3.7
90	3.8
95	4
97	4.1

2. *(16 pts)* Write the matrix A below as a product of elementary matrices. (Find the elementary matrices and indicate clearly the order of the multiplication.)

$$A = \begin{pmatrix} 1 & -2 & -1 \\ -3 & 8 & 7 \\ -4 & 11 & 11 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -2 & -1 \\ -3 & 8 & 7 \\ -4 & 11 & 11 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -2 & -1 \\ 0 & 2 & 4 \\ 0 & 3 & 7 \end{pmatrix} \begin{array}{l} 1 \\ 2 + 3(1) \\ 3 + 4(1) \end{array}$$

$$\begin{pmatrix} 1 & -2 & -1 \\ 0 & 1 & 2 \\ 0 & 3 & 7 \end{pmatrix} \begin{array}{l} 1 \\ 2/2 \\ 3 \end{array}$$

$$\begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{array}{l} 1 + 2(2) \\ 2 \\ 3 - 3(2) \end{array}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{array}{l} 1 - 3(3) \\ 2 - 2(3) \\ 3 \end{array}$$

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad E_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{pmatrix} \qquad E_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{pmatrix}$$

$$E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad E_3^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$E_4 = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad E_4^{-1} = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$E_5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{pmatrix} \qquad E_5^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{pmatrix}$$

$$E_6 = \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad E_6^{-1} = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$E_7 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \qquad E_7^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$E_7 E_6 E_5 E_4 E_3 E_2 E_1 A = I$$

$$\Rightarrow \boxed{A = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1} E_5^{-1} E_6^{-1} E_7^{-1}}$$

3. *(16 pts)* Given the matrix M below, for what vectors $\vec{b} = (b_1, b_2, b_3)$ does the system $M\vec{x} = \vec{b}$ have solutions? In the cases where those solutions exist, find the complete set of those solutions.

$$M = \begin{pmatrix} 1 & -2 & 1 & -4 \\ 2 & -1 & 8 & 1 \\ 3 & -5 & 5 & -9 \end{pmatrix}$$

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & \\ \downarrow & \downarrow & \downarrow & \downarrow & \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & -2 & 1 & -4 & b_1 \\ 2 & -1 & 8 & 1 & b_2 \\ 3 & -5 & 5 & -9 & b_3 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & -2 & 1 & -4 & b_1 \\ 0 & 3 & 6 & 9 & b_2 - 2b_1 \\ 0 & 1 & 2 & 3 & b_3 - 3b_1 \end{array}\right) \begin{array}{l} 1 \\ 2 - 2\,1 \\ 3 - 3\,1 \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & -2 & 1 & -4 & b_1 \\ 0 & 1 & 2 & 3 & b_3 - 3b_1 \\ 0 & 3 & 6 & 9 & b_2 - 2b_1 \end{array}\right) \begin{array}{l} 1 \\ 3 \\ 2 \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 5 & 2 & -5b_1 + 2b_3 \\ 0 & 1 & 2 & 3 & b_3 - 3b_1 \\ 0 & 0 & 0 & 0 & \underbrace{7b_1 + b_2 - 3b_3}_{\text{need}} \end{array}\right) \begin{array}{l} 1 + 2\,2 \\ 2 \\ 3 - 3\,2 \end{array}$$

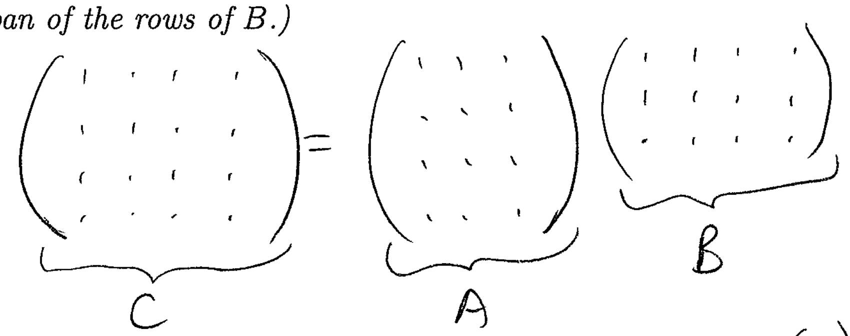
$$\boxed{7b_1 + b_2 - 3b_3 = 0}$$

$$x_1 = -5b_1 + 2b_3 - 5x_3 - 2x_4$$

$$x_2 = -3b_1 + b_3 - 2x_3 - 3x_4$$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \boxed{\begin{pmatrix} -5b_1 + 2b_3 - 5x_3 - 2x_4 \\ -3b_1 + b_3 - 2x_3 - 3x_4 \\ x_3 \\ x_4 \end{pmatrix}}$$

4. *(16 pts)* Suppose that A is a 4×3 matrix and B is a 3×4 matrix. Prove that the 4×4 product matrix $C = AB$ must have determinant equal to zero. *(Hint: Think about the span of the rows of B.)*



The rows of C are all in $RS(B)$; this is 4 vectors in a space of dimension at most 3. So the rows of C are l.d., and thus $\det C = 0$.

5. *(16 pts)* Let $M_{2,2}$ be the vector space of 2×2 matrices with vector addition given by the usual matrix addition, and scalar-vector multiplication given by the usual scalar matrix [redacted] Compute $[A]_\beta$, given the matrix A and basis β below.
multiplication.

$$A = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix} \quad \text{and} \quad \beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \right\}$$

$$\begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix} = c_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + c_2 \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + c_4 \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$$

$$\begin{aligned} c_1 + c_2 &= 2 \\ c_2 &= 1 \\ c_3 + c_4 &= 3 \\ c_4 &= 4 \end{aligned} \implies \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 4 \end{pmatrix}$$

$$\boxed{[A]_\beta = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 4 \end{pmatrix}}$$

6. *(16 pts)* Prove that the collection of vectors $\{v_1, \ldots, v_n\}$ is linearly dependent if and only if at least one of those vectors is a linear combination of the others.

($\Rightarrow$) Assume $\{\vec{v}_1, \ldots, \vec{v}_n\}$ l.d. Then there is a non trivial l.c. with

$$c_1\vec{v}_1 + \ldots + c_n\vec{v}_n = \vec{0}$$

Let c_i be one of the nonzero coefficients. Then

$$\vec{v}_i = \left(\frac{-c_1}{c_i}\right)\vec{v}_1 + \ldots + \left(\frac{-c_n}{c_i}\right)\vec{v}_n$$

(ith term missing)

So one of the vectors ($\vec{v}_i$) is a l.c. of the others.

($\Leftarrow$) Assume one of these vectors ($\vec{v}_i$) is a l.c. of the others.

$$\vec{v}_i = k_1\vec{v}_1 + \ldots + k_n\vec{v}_n$$

(ith term missing)

Then

$$k_1\vec{v}_1 + \ldots + (-1)\vec{v}_i + \ldots k_n\vec{v}_n = \vec{0}$$

This is a non trivial l.c. that equals $\vec{0}$, so

$\{\vec{v}_1, \ldots, \vec{v}_n\}$ is l.d.

7. *(16 pts)* Suppose we know that A is a 3×3 matrix and that

$$A\begin{bmatrix}1\\2\\1\end{bmatrix} = \begin{bmatrix}1\\5\\2\end{bmatrix} \quad \text{and} \quad A\begin{bmatrix}3\\1\\3\end{bmatrix} = \begin{bmatrix}0\\2\\1\end{bmatrix} \quad \text{and} \quad A\begin{bmatrix}2\\4\\4\end{bmatrix} = \begin{bmatrix}2\\3\\1\end{bmatrix}$$

Without computing the matrix A itself, compute the determinant of A. *(Hint: Find a way to write $AB = C$ where you can compute the determinants of B and C.)*

Let $B = \begin{pmatrix}1 & 3 & 2\\2 & 1 & 4\\1 & 3 & 4\end{pmatrix}$, $C = \begin{pmatrix}1 & 0 & 2\\5 & 2 & 3\\2 & 1 & 1\end{pmatrix}$

Then the three given statements are equivalent to

$$AB = C$$

So we have $(\det A)(\det B) = (\det C)$

and $\det B = (1)(-8) - (3)(4) + (2)(5) = -10$

$\det C = (1)(-1) - (0)(-1) + (2)(1) = 1$

So $\boxed{\det A = \frac{-1}{10}}$