EXAM 3

Math 107, 2009-2010 Spring, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.
All answers must be simplified. All of the policies and guidelines
on the class webpages are in effect on this exam.

Good luck!

Name Solutions

ID number____________________

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ____________________

1. __________

2. __________

3. __________

4. __________

5. __________

Total Score __________ (/100 points)

1. *(20 pts)* Find the matrix A that satisfies the equations

$$A\begin{bmatrix}0\\2\\1\end{bmatrix}=\begin{bmatrix}-2\\-3\\0\end{bmatrix} \quad\text{and}\quad A\begin{bmatrix}2\\3\\0\end{bmatrix}=\begin{bmatrix}-1\\-4\\-1\end{bmatrix} \quad\text{and}\quad A\begin{bmatrix}1\\4\\1\end{bmatrix}=\begin{bmatrix}0\\-2\\-1\end{bmatrix}$$

Choose $[\vec{v}_1]_{\mathcal{S}}=\begin{pmatrix}0\\2\\1\end{pmatrix}$, $[\vec{v}_2]_{\mathcal{S}}=\begin{pmatrix}2\\3\\0\end{pmatrix}$, $[\vec{v}_3]_{\mathcal{S}}=\begin{pmatrix}1\\4\\1\end{pmatrix}$, $\mathcal{V}=\{\vec{v}_1,\vec{v}_2,\vec{v}_3\}$

Then $T(\vec{v}_1)=-\vec{v}_2$, $T(\vec{v}_2)=-\vec{v}_3$, $T(\vec{v}_3)=-\vec{v}_1$

and $[T]_{\mathcal{V}}^{\mathcal{V}}=\begin{pmatrix}0&0&-1\\-1&0&0\\0&-1&0\end{pmatrix}$

Then $A=[T]_{\mathcal{S}}^{\mathcal{S}}=[I]_{\mathcal{V}}^{\mathcal{S}}[T]_{\mathcal{V}}^{\mathcal{V}}[I]_{\mathcal{S}}^{\mathcal{V}}$

$$=\begin{pmatrix}0&2&1\\2&3&4\\1&0&1\end{pmatrix}\begin{pmatrix}0&0&-1\\-1&0&0\\0&-1&0\end{pmatrix}\begin{pmatrix}0&2&1\\2&3&4\\1&0&1\end{pmatrix}^{-1}$$

We find the inverse:

$$\left(\begin{array}{ccc|ccc}0&2&1&1&0&0\\2&3&4&0&1&0\\1&0&1&0&0&1\end{array}\right)$$

$$\left(\begin{array}{ccc|ccc}1&0&1&0&0&1\\0&2&1&1&0&0\\2&3&4&0&1&0\end{array}\right)\begin{array}{l}3\\1\\2\end{array}$$

$$\left(\begin{array}{ccc|ccc}1&0&1&0&0&1\\0&2&1&1&0&0\\0&3&2&0&1&-2\end{array}\right)\begin{array}{l}1\\2\\3-2(1)\end{array}$$

$$\left(\begin{array}{ccc|ccc}1&0&1&0&0&1\\0&2&1&1&0&0\\0&1&1&-1&1&-2\end{array}\right)\begin{array}{l}1\\2\\3-2\end{array}$$

$$\left(\begin{array}{ccc|ccc}1&0&1&0&0&1\\0&1&1&-1&1&-2\\0&2&1&1&0&0\end{array}\right)\begin{array}{l}1\\3\\2\end{array}$$

$$\left(\begin{array}{ccc|ccc}1&0&1&0&0&1\\0&1&1&-1&1&-2\\0&0&-1&3&-2&4\end{array}\right)\begin{array}{l}1\\2\\3-2(2)\end{array}$$

$$\left(\begin{array}{ccc|ccc}1&0&0&3&-2&5\\0&1&0&2&-1&2\\0&0&1&-3&2&-4\end{array}\right)\begin{array}{l}1+3\\2+3\\-3\end{array}$$

Multiplying the above, we get

$$A=\begin{pmatrix}-8&5&-12\\-11&6&-15\\1&-1&2\end{pmatrix}$$

2. *(20 pts)* Prove that if A is an orthogonal matrix, then $A^T = A^{-1}$.

If A is orthogonal, then

$$A = \begin{pmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{pmatrix}$$

where $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is an orthonormal basis.

Then

$$A^T A = \begin{pmatrix} \text{---} & \vec{v}_1 & \text{---} \\ & \vdots & \\ \text{---} & \vec{v}_n & \text{---} \end{pmatrix} \begin{pmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{pmatrix}$$

The elements of this product are

$$m_{ij} = \vec{v}_i \cdot \vec{v}_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

so we have

$$A^T A = I$$

and thus $A^T = A^{-1}$.

3. *(20 pts)* Find a fundamental set of solutions to the system of equations

$$\vec{y}\,' = A\vec{y} \quad \text{where} \quad A = \begin{pmatrix} 0 & -4 & 2 \\ -1 & -5 & 2 \\ -4 & -23 & 9 \end{pmatrix}$$

(You may use the fact that the characteristic polynomial of A is $p(\lambda) = (\lambda-1)^2(\lambda-2)$. Two vectors of interest might be $(2,0,1)$ and $(3,1,5)$.)

Note $A\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ and $A\begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ 10 \end{pmatrix} = 2\begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix}$

So these are eigenvectors for the eigenvalues 1, 2 (resp.).

To determine if there is another eigenvector for $\lambda = 1$:

$$A - 1I = \begin{pmatrix} -1 & -4 & 2 \\ -1 & -6 & 2 \\ -4 & -23 & 8 \end{pmatrix}$$ $\leftarrow$ This has rank = 2, so its NS has dim 1, so there is only 1 evect. for $\lambda = 1$.

We find the other Jordan basis vector by solving

$$(A - 1I)\vec{v}_2 = \vec{v}_1$$

$$\left(\begin{array}{ccc|c} -1 & -4 & 2 & 2 \\ -1 & -6 & 2 & 0 \\ -4 & -23 & 8 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 4 & -2 & -2 \\ 0 & -2 & 0 & -2 \\ 0 & -7 & 0 & -7 \end{array}\right) \begin{array}{l} -① \\ ② - ① \\ ③ - 4① \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -2 & -6 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} ① + 2② \\ ②/-2 \\ ③ - \tfrac{7}{2}② \end{array}$$

$$\Rightarrow \vec{v}_2 = \begin{pmatrix} -6+2z \\ 1 \\ z \end{pmatrix}$$

Choosing $z = 0$, we have a Jordan basis

$$\mathcal{V} = \left\{ \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -6 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix} \right\}$$

and Jordan form

$$J = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

(over)

Choosing this basis, we have solutions

$$\left\{ e^{xA}\vec{v}_1 \ , \ e^{xA}\vec{v}_2 \ , \ e^{xA}\vec{v}_3 \right\}$$

with

$$e^{xA}\vec{v}_1 = e^{1x}\vec{v}_1 = \begin{pmatrix} 2e^x \\ 0 \\ e^x \end{pmatrix}$$

$$e^{xA}\vec{v}_2 = e^{1x}\left(\vec{v}_2 + x\vec{v}_1\right) = \begin{pmatrix} -6e^x + 2xe^x \\ e^x \\ xe^x \end{pmatrix}$$

$$e^{xA}\vec{v}_3 = e^{2x}\vec{v}_3 = \begin{pmatrix} 3e^{2x} \\ e^{2x} \\ 5e^{2x} \end{pmatrix}$$

So our fundamental set is

$$\left\{ \begin{pmatrix} 2e^x \\ 0 \\ e^x \end{pmatrix}, \begin{pmatrix} -6e^x + 2xe^x \\ e^x \\ xe^x \end{pmatrix}, \begin{pmatrix} 3e^{2x} \\ e^{2x} \\ 5e^{2x} \end{pmatrix} \right\}$$

4. *(20 pts)* Find the general solution to the system of equations below.

$$\begin{aligned} y_1' &= -y_1 + 2y_2 + 4 \\ y_2' &= -6y_1 + 6y_2 + \sin(x) \end{aligned}$$

The homogeneous system is

$$\vec{y}\,' = \underbrace{\begin{pmatrix} -1 & 2 \\ -6 & 6 \end{pmatrix}}_{A} \vec{y}$$

The char. poly is $p(\lambda) = (\lambda+1)(\lambda-6) - (-12) = \lambda^2 - 5\lambda + 6 = (\lambda-2)(\lambda-3)$

$\lambda = 2$: $A - \lambda I = \begin{pmatrix} -3 & 2 \\ -6 & 4 \end{pmatrix}$, has $NS = \text{span}\left\{ \begin{pmatrix} 2 \\ 3 \end{pmatrix} \right\}$

$\lambda = 3$: $A - \lambda I = \begin{pmatrix} -4 & 2 \\ -6 & 3 \end{pmatrix}$, has $NS = \text{span}\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$

Eigenbasis is $\left\{ \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$

Fund. sols: $\left\{ \begin{pmatrix} 2e^{2x} \\ 3e^{2x} \end{pmatrix}, \begin{pmatrix} e^{3x} \\ 2e^{3x} \end{pmatrix} \right\}$

<u>Particular solutions:</u>

For $\vec{y}\,' = A\vec{y} + \begin{pmatrix} 4 \\ 0 \end{pmatrix}$, we try $\vec{y}_{p_1} = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = A\vec{y}_{p_1} + \begin{pmatrix} 4 \\ 0 \end{pmatrix}$$

$$\vec{y}_{p_1} = A^{-1}\begin{pmatrix} -4 \\ 0 \end{pmatrix} = \frac{\begin{pmatrix} 6 & -2 \\ 6 & -1 \end{pmatrix}}{6} \begin{pmatrix} -4 \\ 0 \end{pmatrix} = \begin{pmatrix} -4 \\ -4 \end{pmatrix}$$

For $\vec{z}' = A\vec{z} + \begin{pmatrix} 0 \\ e^{ix} \end{pmatrix}$, try $\vec{z}_p = e^{ix}\vec{v}$

$$ie^{ix}\vec{v} = Ae^{ix}\vec{v} + e^{ix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$(A - iI)\vec{v} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$\vec{v} = (A - iI)^{-1}\begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -1-i & 2 \\ -6 & 6-i \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$= \frac{\begin{pmatrix} 6-i & -2 \\ 6 & -1-i \end{pmatrix}}{5-5i}\begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1+i \end{pmatrix}\frac{1+i}{10}$$

$$= \begin{pmatrix} (1+i)/5 \\ i/5 \end{pmatrix}$$

$$\vec{z}_p = e^{ix}\begin{pmatrix} \frac{1+i}{5} \\ i/5 \end{pmatrix} = (\cos x + i\sin x)\begin{pmatrix} \frac{1+i}{5} \\ \frac{i}{5} \end{pmatrix}$$

For $\vec{y}' = A\vec{y} + \text{Im}\begin{pmatrix} 0 \\ e^{ix} \end{pmatrix}$ then, we take $\vec{y}_{p_2} = \text{Im}(\vec{z}_p)$

$$\vec{y}_{p_2} = \begin{pmatrix} \frac{\cos x + \sin x}{5} \\ \frac{\cos x}{5} \end{pmatrix}$$

Then our general solution is

$$\vec{y} = c_1\begin{pmatrix} 2e^{2x} \\ 3e^{2x} \end{pmatrix} + c_2\begin{pmatrix} e^{3x} \\ 2e^{3x} \end{pmatrix} + \begin{pmatrix} -4 \\ -4 \end{pmatrix} + \begin{pmatrix} \frac{\cos x + \sin x}{5} \\ \frac{\cos x}{5} \end{pmatrix}$$

5. *(20 pts)* Find a first order system of linear differential equations whose solution would also give the solution to the system below. Indicate clearly the relationship between the functions described in each system.

$$\begin{aligned} y_1''' &= 2y_1'' - y_2 + 3y_2' \\ y_2''' &= 4y_1 + 2y_2' - y_2'' \end{aligned}$$

Let
$$\begin{aligned} y_1 &= u_1 \\ y_1' &= u_2 \\ y_1'' &= u_3 \quad \longleftarrow \quad \text{so } y_1''' = u_3' \\ y_2 &= u_4 \\ y_2' &= u_5 \\ y_2'' &= u_6 \quad \longleftarrow \quad \text{so } y_2''' = u_6' \end{aligned}$$

Then we have

$$\begin{aligned} u_1' &= 0u_1 + 1u_2 + 0u_3 + 0u_4 + 0u_5 + 0u_6 \\ u_2' &= 0u_1 + 0u_2 + 1u_3 + 0u_4 + 0u_5 + 0u_6 \\ u_3' &= 0u_1 + 0u_2 + 2u_3 - 1u_4 + 3u_5 + 0u_6 \\ u_4' &= 0u_1 + 0u_2 + 0u_3 + 0u_4 + 1u_5 + 0u_6 \\ u_5' &= 0u_1 + 0u_2 + 0u_3 + 0u_4 + 0u_5 + 1u_6 \\ u_6' &= 4u_1 + 0u_2 + 0u_3 + 0u_4 + 2u_5 - 1u_6 \end{aligned}$$

Or

$\vec{u}' = A\vec{u}$, with

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 4 & 0 & 0 & 0 & 2 & -1 \end{pmatrix}$$