EXAM 2

Math 107, 2009-2010 Spring, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

Good luck!

Name Solutions

ID number____________________

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ____________________

1. __________
2. __________
3. __________
4. __________
5. __________
6. __________

Total Score __________ (/100 points)

1. *(15 pts)* Prove or disprove: $V = \left\{ f \in C^3[a,b] \;\middle|\; \int_a^b f(x)\,dx + f'''(m) = 0 \text{ (where } m \in (a,b)) \right\}$ is a subspace of $C^3[a,b]$.

Say $f_1, f_2 \in V$. Then

$$\int_a^b (f_1+f_2)\,dx + (f_1+f_2)'''(m) = \left(\int_a^b f_1\,dx + f_1'''(m)\right) + \left(\int_a^b f_2\,dx + f_2'''(m)\right)$$

$$= 0 + 0 = 0$$

$$\int_a^b (cf_1)\,dx + (cf_1)'''(m) = c\left(\int_a^b f_1\,dx + f_1'''(m)\right) = c \cdot 0 = 0$$

So V is closed under addition and scalar mult., so it is a subspace

2. *(15 pts)* Determine if the following functions are independent or dependent:

$$f_1(x) = \sin\left(x + \frac{\pi}{6}\right) \qquad f_2(x) = \sin\left(x + \frac{\pi}{4}\right) \qquad f_3(x) = \sin\left(x + \frac{\pi}{3}\right)$$

$$f_1 = \sin x \cos\tfrac{\pi}{6} + \cos x \sin\tfrac{\pi}{6} = \tfrac{\sqrt{3}}{2}\sin x + \tfrac{1}{2}\cos x$$

$$f_2 = \sin x \cos\tfrac{\pi}{4} + \cos x \sin\tfrac{\pi}{4} = \tfrac{\sqrt{2}}{2}\sin x + \tfrac{\sqrt{2}}{2}\cos x$$

$$f_3 = \sin x \cos\tfrac{\pi}{3} + \cos x \sin\tfrac{\pi}{3} = \tfrac{1}{2}\sin x + \tfrac{\sqrt{3}}{2}\cos x$$

$$f_1 + f_3 = \left(\frac{1+\sqrt{3}}{2}\right)(\sin x + \cos x) = \left(\frac{1+\sqrt{3}}{\sqrt{2}}\right) f_2$$

So these functions are dependent.

Alt.: These functions are all solutions to $y'' + y = 0$, and this has a 2-dimensional space of solutions. Three vectors in a 2-d subspace must be dependent. So these functions are dependent.

3. *(15 pts)* A bath has a constant temperature of 40°C, and at time $t = 0$ a metal rod is immersed in the bath. The initial temperature of the rod is 13°C, and after one minute the temperature is 22. Assuming that the temperature follows Newton's law of cooling, find the temperature as a function of time, and determine what time the rod will reach 32°C.

T = temperature of the rod

$u = 40 - T$ = temperature difference

Newton's law: $\quad u' = ku$

$$\Rightarrow u = Ae^{-kt}$$

Initially: $\quad T(0) = 13 \quad \Rightarrow u(0) = 27$

$$\Rightarrow u = 27e^{-kt}$$

At $t=1$: $\quad T(1) = 22 \quad \Rightarrow u(1) = 18$

$$\Rightarrow 18 = 27e^{-k}$$

$$\Rightarrow e^{-k} = \frac{2}{3}$$

$$\Rightarrow \boxed{u = 27\left(\frac{2}{3}\right)^t}$$

When $T = 32$: $\quad \Rightarrow u = 8$

$$\Rightarrow 8 = 27\left(\frac{2}{3}\right)^t$$

$$\Rightarrow \frac{8}{27} = \left(\frac{2}{3}\right)^t$$

$$\Rightarrow \boxed{t = 3}$$

4. *(20 pts)* Find a fundamental set of solutions to the differential equation

$$y''' - 11y'' + 26y' - 16y = 0$$

(Hint: One of the solutions is $y(x) = e^{8x}$*.)*

$p(\lambda) = \lambda^3 - 11\lambda^2 + 26\lambda - 16$ (Given: 8 is a root.)

$$\begin{array}{r|l} & \lambda^2 - 3\lambda + 2 \\ \hline \lambda - 8 & \lambda^3 - 11\lambda^2 + 26\lambda - 16 \\ & \underline{\lambda^3 - 8\lambda^2} \\ & -3\lambda^2 + 26\lambda - 16 \\ & \underline{-3\lambda^2 + 24\lambda} \\ & 2\lambda - 16 \\ & \underline{2\lambda - 16} \\ & 0 \end{array}$$

So $p(\lambda) = (\lambda-8)(\lambda^2-3\lambda+2) = (\lambda-8)(\lambda-1)(\lambda-2)$

$\Rightarrow e^{8x}, e^{1x}, e^{2x}$ are solutions.

Wronskian:

$$W(0) = \det\begin{pmatrix} 1 & 1 & 1 \\ 8 & 1 & 2 \\ 64 & 1 & 4 \end{pmatrix}$$

$$= 1(2) - 1(128-32) + 1(-56)$$

$$= -150 \neq 0 \Rightarrow \text{independent}$$

So $\{e^{8x}, e^{1x}, e^{2x}\}$ is a fundamental set of sol's.

5. *(20 pts)* A forced and damped oscillation is described by the differential equation

$$u'' + 3u' + 2u = \cos(3t)$$

Compute (directly, without using a memorized formula) the gain in this system (the ratio of the amplitudes of the forcing term and the particular solution).

$$\cos(3t) = \mathrm{Re}\left(e^{3it}\right)$$

So $$u = \mathrm{Re}(z)$$

where $$z'' + 3z' + 2z = e^{3it}$$

We know $z = Te^{3it}$ will be a particular solution.

So $$(-9)Te^{3it} + 3(3i)Te^{3it} + 2Te^{3it} = e^{3it}$$

$$\Rightarrow T(-9 + 9i + 2) = 1$$

$$\Rightarrow T = \frac{1}{-7+9i} = \frac{7+9i}{-49-81} = -\frac{7+9i}{130}$$

$$= Ge^{-i\phi}, \text{ with } G = \frac{1}{\sqrt{130}}$$

$$\Rightarrow z = \frac{1}{\sqrt{130}} e^{i(3t-\phi)} = \frac{1}{\sqrt{130}} \cos(3t-\phi) + \frac{i}{\sqrt{130}} \sin(3t-\phi)$$

$$\Rightarrow u = \frac{1}{\sqrt{130}} \cos(3t-\phi)$$

So the gain is $\frac{1}{\sqrt{130}}$.

6. *(15 pts)* Find a particular solution to the differential equation

$$y'' + 4y = \sin(2x)$$

$p(\lambda) = \lambda^2 + 4 = (\lambda + 2i)(\lambda - 2i)$

$g(x) = e^{ax}\sin(bx)$ with $r = a + bi = 2i$ ← root of p, mult. $= 1$

So there will be a solution of the form

$$y = Ax\sin 2x + Bx\cos 2x$$

Differentiating gives us

$$y' = A\sin 2x + 2Ax\cos 2x + B\cos 2x - 2Bx\sin 2x$$

$$y'' = 2A\cos 2x + 2A\cos 2x - 4Ax\sin 2x - 2B\sin 2x - 2B\sin 2x - 4Bx\cos 2x$$

Then the equation becomes

$$(4A\cos 2x - 4B\sin 2x - \cancel{4Ax\sin 2x} - \cancel{4Bx\cos 2x}) + 4(\cancel{Ax\sin 2x} + \cancel{Bx\cos 2x}) = \sin 2x$$

$$4A\cos 2x - 4B\sin 2x = \sin 2x$$

$$\Rightarrow -4B = 1, \quad 4A = 0$$

$$\Rightarrow A = 0, \quad B = -1/4$$

So $\boxed{y = -\frac{1}{4}x\cos 2x}$