# EXAM 1

Math 107, 2009-2010 Spring, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

Good luck!

Name Solutions

ID number ____________

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ____________

1. ____________

2. ____________

3. ____________

4. ____________

5. ____________

6. ____________

7. ____________

Total Score ____________ (/100 points)

1. *(15 pts)* Consider the matrices $A$ and $B$ defined by

$$A = \begin{pmatrix} 1 & 17 & 6 & 7 \\ 4 & 2 & 8 & 4 \\ 5 & 8 & 10 & 2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 7 & 1 & 0 & -3 \\ -2 & 5 & 0 & 2 \\ -5 & 7 & 0 & 10 \end{pmatrix}$$

Find a matrix $C$ for which $A = CB$, or show that such a matrix $C$ does not exist.

If $C$ were to exist, then each row of $A$ would be a linear combination of the rows of $B$. All of $B$'s rows have 0 in the 3rd position, so the every l.c. must also, and thus also the rows of $A$ must have 0 in the 3rd position.

This is not the case, so $C$ cannot exist.

2. *(15 pts)* Find the inverse of the matrix $A$ defined by

$$A = \begin{pmatrix} 3 & -4 & 7 \\ 0 & 0 & -1 \\ -2 & 3 & 4 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 3 & -4 & 7 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ -2 & 3 & 4 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 3 & -4 & 7 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 1 & -1 & 11 & 1 & 0 & 1 \end{array}\right) \begin{array}{l} (1) \\ (2) \\ (3) + (1) \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 11 & 1 & 0 & 1 \\ 3 & -4 & 7 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \end{array}\right) \begin{array}{l} (3) \\ (1) \\ (2) \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 11 & 1 & 0 & 1 \\ 0 & -1 & -26 & -2 & 0 & -3 \\ 0 & 0 & -1 & 0 & 1 & 0 \end{array}\right) \begin{array}{l} (1) \\ (2) - 3(1) \\ (3) \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 37 & 3 & 0 & 4 \\ 0 & -1 & -26 & -2 & 0 & -3 \\ 0 & 0 & -1 & 0 & 1 & 0 \end{array}\right) \begin{array}{l} (1) - (2) \\ (2) \\ (3) \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 37 & 4 \\ 0 & -1 & 0 & -2 & -26 & -3 \\ 0 & 0 & -1 & 0 & 1 & 0 \end{array}\right) \begin{array}{l} (1) + 37(3) \\ (2) - 26(3) \\ (3) \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 37 & 4 \\ 0 & 1 & 0 & 2 & 26 & 3 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{array}\right) \begin{array}{l} (1) \\ -(2) \\ -(3) \end{array}$$

$$A^{-1} = \begin{pmatrix} 3 & 37 & 4 \\ 2 & 26 & 3 \\ 0 & -1 & 0 \end{pmatrix}$$

3. *(15 pts)* Use permutations to compute the determinant of the matrix $M$ given below. (Do NOT use another method.)

$$M = \begin{pmatrix} 4 & 1 & -1 \\ 3 & 2 & 2 \\ 7 & 5 & 0 \end{pmatrix}$$

| | | | |
|---|---|---|---|
| $1 \to 1$, $2 \to 2$, $3 \to 3$ | $\begin{pmatrix} \cdot & & \\ & \cdot & \\ & & \cdot \end{pmatrix}$ | (even) | $(+)(4)(2)(0)$ |
| $1 \to 2$, $2 \to 1$, $3 \to 3$ | $\begin{pmatrix} & \cdot & \\ \cdot & & \\ & & \cdot \end{pmatrix}$ | (odd) | $(-)(1)(3)(0)$ |
| $1 \to 3$, $2 \to 2$, $3 \to 1$ | $\begin{pmatrix} & & \cdot \\ & \cdot & \\ \cdot & & \end{pmatrix}$ | (odd) | $(-)(-1)(2)(7)$ |
| $1 \to 1$, $2 \to 3$, $3 \to 2$ | $\begin{pmatrix} \cdot & & \\ & & \cdot \\ & \cdot & \end{pmatrix}$ | (odd) | $(-)(4)(2)(5)$ |
| $1 \to 2$, $2 \to 3$, $3 \to 1$ | $\begin{pmatrix} & \cdot & \\ & & \cdot \\ \cdot & & \end{pmatrix}$ | (even) | $(+)(1)(2)(7)$ |
| $1 \to 3$, $2 \to 1$, $3 \to 2$ | $\begin{pmatrix} & & \cdot \\ \cdot & & \\ & \cdot & \end{pmatrix}$ | (even) | $(+)(-1)(3)(5)$ |

$= 0 + 0 + 14 - 40 + 14 - 15$

$= -27$

4. *(15 pts)* Use pivots to show that it is not possible for any collection of 4 vectors to span $\mathbb{R}^5$. (Make sure to explain all of the steps in your reasoning.)

To span $\mathbb{R}^5$, need $c_1\vec{v}_1 + \dots + c_4\vec{v}_4 = \vec{b}$ to have solutions for all $\vec{b} \in \mathbb{R}^5$.

So $A\vec{c} = \vec{b}$ must have solutions for all $\vec{b}$.

So $A = \begin{pmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_4 \\ | & & | \end{pmatrix}$ must have a pivot in every row.

But, $A$ has only 4 columns, so rank $(A) \leq 4$.

So $A$ cannot have a pivot in every row, and thus these vectors cannot span $\mathbb{R}^5$.

5. *(15 pts)* Show that in any vector space $V$, the zero vector $\vec{0}$ is unique.

Suppose there are two zero vectors, $\vec{0}_1$ and $\vec{0}_2$.

Note $\vec{0}_1 + \vec{0}_2 = \vec{0}_2$, because $\vec{0}_1$ is a zero vector.

Also $\vec{0}_1 + \vec{0}_2 = \vec{0}_1$, because $\vec{0}_2$ is a zero vector.

Then we have $\vec{0}_1 = \vec{0}_2$. So, the zero vector must be unique.

6. *(10 pts)* We will call a square matrix "spinny" if rotating the matrix by a quarter turn does not change the matrix. For example, the matrix $A$ below is spinny:

$$A = \begin{pmatrix} 1 & 3 & 2 & 1 \\ 2 & 0 & 0 & 3 \\ 3 & 0 & 0 & 2 \\ 1 & 2 & 3 & 1 \end{pmatrix}$$

Show that every $3 \times 3$ spinny matrix must have determinant zero. Is this also true for $4 \times 4$ spinny matrices? Prove or find a counterexample.

If $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ is spinny, then the corners must be equal: $a_{11} = a_{13} = a_{33} = a_{31}$.

Also, the edges must equal: $a_{12} = a_{23} = a_{32} = a_{21}$

So we must have $A = \begin{pmatrix} b & c & b \\ c & d & c \\ b & c & b \end{pmatrix}$ for some values $b, c, d$. The 1st and 3rd rows are equal, so $\det A = 0$.

This is <u>not</u> true for $4 \times 4$ matrices. The given example matrix above has determinant (expand along 2nd row)

$$(-)(2)\det\begin{pmatrix} 3 & 2 & 1 \\ 0 & 0 & 2 \\ 2 & 3 & 1 \end{pmatrix} + (+)(3)\det\begin{pmatrix} 1 & 3 & 2 \\ 3 & 0 & 0 \\ 1 & 2 & 3 \end{pmatrix}$$

$$= (-2)(-10) + (3)(-15)$$

$$= -65 \neq 0.$$

7. *(15 pts)* Write the matrix $A$ below as a product of elementary matrices.

$$A = \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 \\ 0 & -4 \end{pmatrix} \begin{matrix} ① \\ ② - 2① \end{matrix} \qquad E_1 = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \begin{matrix} ① \\ ②/-4 \end{matrix} \qquad E_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1/4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{matrix} ① - 3② \\ ② \end{matrix} \qquad E_3 = \begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix}$$

The row reduction can be written as

$$E_3 E_2 E_1 A = I$$

So

$$A = E_1^{-1} E_2^{-1} E_3^{-1}$$

$$\boxed{A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -4 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}}$$