# EXAM 3

Math 212, 2023 Summer Term 2, Clark Bray.

Name:___Solutions_____    Section:_____    Student ID:_____

## GENERAL RULES

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING TO RECEIVE CREDIT.
CLARITY WILL BE CONSIDERED IN GRADING.

No notes, no books, no calculators.

All answers must be reasonably simplified.

All of the policies and guidelines on the class webpages are in effect on this exam.

## WRITING RULES

Do not remove the staple, tear pages out of the staple, or tamper with the exam packet in any way. Do not write anything near the staple – this may be cut off.

Use black pen only. You may use a pencil for initial sketches of diagrams, but the final sketch must be drawn over in black pen and you must wipe all erasure residue from the paper.

Work for a given question can be done ONLY on the front or back of the page the question is written on. Room for scratch work is available on the back of this cover page, and on the two blank pages at the end of this packet; scratch work will NOT be graded.

## DUKE COMMUNITY STANDARD STATEMENT

"I have adhered to the Duke Community Standard in completing this examination."
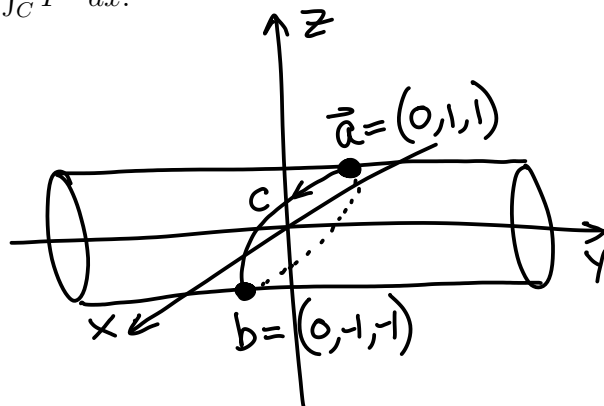
Signature: _____

1. *(20 pts)* The curve $I$ is the intersection of the cylinder $x^2 + z^2 = 1$ with the plane $y = z$. The curve $C$ is the portion of $I$ with $x \geq 0$, oriented downward. The vector field $F$ is defined by $\vec{F}(x, y, z) = (e^{z^2}, ze^y, 2xze^{z^2} + e^y)$. Compute $\int_C \vec{F} \cdot d\vec{x}$.

$$\nabla \times \vec{F} = \left( e^y - e^y, \; 2ze^{z^2} - 2ze^{z^2}, \; 0 - 0 \right)$$

$$= \vec{0}$$

So $\vec{F}$ is a gradient.


$\vec{a} = (0,1,1)$
$b = (0,-1,-1)$

$$\int P\,dx = \int e^{z^2} dx \qquad = xe^{z^2} \qquad + \boxed{c_1(y,z)}$$

$$f = \int Q\,dy = \int ze^y\,dy \qquad = \qquad ze^y + \boxed{c_2(x,z)}$$

$$\int R\,dz = \int 2xze^{z^2} + e^y\,dz \; = \; xe^{z^2} + ze^y + \boxed{c_3(x,y)}$$

We can choose: $\qquad f = \boxed{xe^{z^2}} + \boxed{ze^y}$

$$\text{Then } \int_C \vec{F} \cdot d\vec{x} = \int_C \nabla f \cdot d\vec{x}$$

$$= f(\vec{b}) - f(\vec{a})$$

$$= f(0,-1,-1) - f(0,1,1)$$

$$= \left( -1 e^{-1} \right) - \left( 1 e^1 \right)$$

$$= \frac{-1}{e} - e$$

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2. *(20 pts)* The surface $S$ is the portion of $z = x^2$ over the rectangle $R = [0,1] \times [1,2]$ in the $xy$-plane, oriented in the positive $x$-direction. The vector field $\vec{F}$ is defined by $\vec{F}(x,y,z) = (0,\ 1,\ z)$. Compute $\iint_S \vec{F} \cdot d\vec{S}$.

We parametrize $S$ by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} u \\ v \\ u^2 \end{pmatrix} \quad \text{with } (u,v) \in \left( P = [0,1] \times [1,2] \right) \text{ in the } uv\text{-plane.}$$

$$\Rightarrow \vec{X}_u = \begin{pmatrix} 1 \\ 0 \\ 2u \end{pmatrix}, \quad \vec{X}_v = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \Rightarrow \vec{N} = \begin{pmatrix} -2u \\ 0 \\ 1 \end{pmatrix}$$

oriented the wrong way!

Then $\iint_S \vec{F} \cdot d\vec{S} = -\iint_P \vec{F} \cdot \vec{N}\, du\, dv$

$$= -\iint_P \begin{pmatrix} 0 \\ 1 \\ z \end{pmatrix} \cdot \begin{pmatrix} -2u \\ 0 \\ 1 \end{pmatrix} du\, dv$$

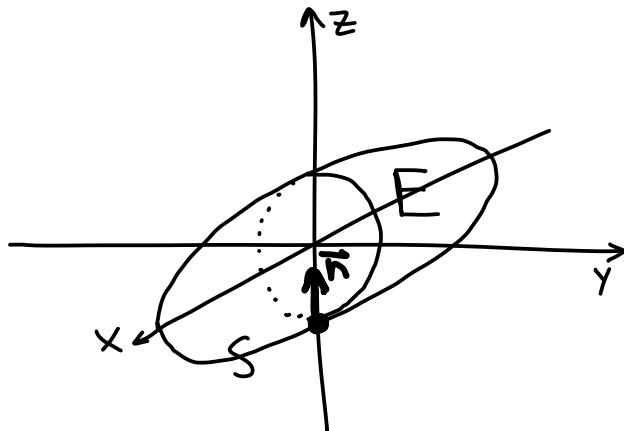$$= -\iint_P \begin{pmatrix} 0 \\ 1 \\ u^2 \end{pmatrix} \cdot \begin{pmatrix} -2u \\ 0 \\ 1 \end{pmatrix} du\, dv$$

$$= -\int_1^2 \int_0^1 u^2\, du\, dv = \frac{-1}{3}$$

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3. *(20 pts)* The surface $S$ has equation $x^2 + 4y^2 + 9z^2 = 36$, and at $(0, -3, 0)$ we have $\vec{n} = (0, 1, 0)$. Compute the flux through $S$ of the vector field $\vec{F}(x, y, z) = (x^2 - e^z, \ x - 4y + \sin z, \ x^3 y^3 - 4)$.

S bounds the solid ellipsoid E and is oriented inward. So

$$S = -\partial E$$

By the divergence theorem we have

$$\iint_S \vec{F} \cdot d\vec{S} = -\iiint_E \nabla \cdot \vec{F} \, dV$$

$$= -\iiint_E (2x - 4) \, dV$$

$$= \underbrace{\iiint_E (-2x) \, dV}_{\substack{=0 \text{ by symmetry} \\ \text{through yz-plane}}} + \iiint_E 4 \, dV$$

$$= 4 \left( \text{vol. of E} \right)$$

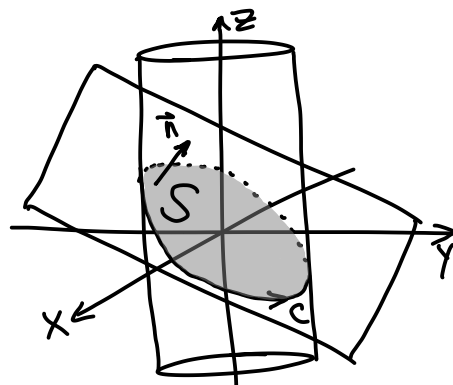$$= 4 \left( \frac{4}{3} \pi (6 \cdot 3 \cdot 2) \right)$$

$$= 192 \pi$$

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4. *(20 pts)*

(a) The curve $C$ is parametrized by $\vec{x}(t) = (\cos t, \sin t, \cos t - 3 \sin t)$. Compute the line integral along $C$ of the vector field $\vec{F} = (x^3 + z, x - e^{y^3}, 4y + \sin(z^3))$. *(Hint: Find two familiar equations that the parametrization satisfies.)*

The parametrization satisfies
$x^2 + y^2 = 1$ and $z = x - 3y$, so
$C$ is that intersection oriented
CCW as seen from above.

$C$ is the boundary of $S$, the part of the plane
inside the cylinder, oriented upward. Then by Stokes,

$$\int_C \vec{F} \cdot d\vec{x} = \int_{\partial S} \vec{F} \cdot d\vec{x} = \iint_S (\nabla \times \vec{F}) \cdot \vec{n} \, dS$$

$$= \iint_S \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} \frac{1}{\sqrt{11}} \, dS = 0$$

(b) In $\mathbb{R}^2$, we have the vector field $\vec{F}(x,y) = (-y^3, x^3)$. A very small circle $C$ of known radius $a$, oriented counterclockwise, can be placed with its center at any point $\vec{c} = (x_0, y_0)$. Use Green's theorem and an approximation to find the point $\vec{c}$ that will minimize the circulation $\int_C \vec{F} \cdot d\vec{x}$.

$$\text{grn} \vec{F} = (3x^2) - (-3y^2) = 3(x^2 + y^2)$$

This is $\sim$constant over very small disks $D$ bounded by
such circles as $C$ centered at $\vec{c} = (x_0, y_0)$, so

$$\int_C \vec{F} \cdot d\vec{x} = \iint_D \text{grn} \vec{F} \, dA \cong (3(x_0^2 + y_0^2))(\text{area of } D)$$

$$\cong (3(x_0^2 + y_0^2))(\pi a^2)$$

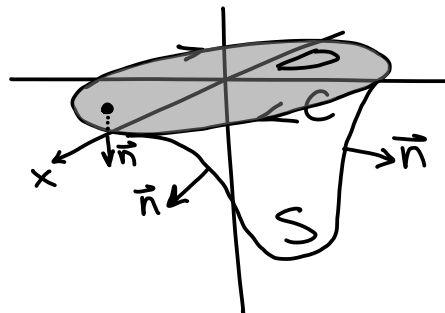This is minimized when $\vec{c} = (x_0, y_0) = \vec{0}$.

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5. *(20 pts)* The surface $S$ is the portion of $z = (x^2+y^2-1)(e^{xy})$ that is below the $xy$-plane, oriented downward. Compute the flux through $S$ of the vector field $\vec{F}(x,y,z) = (3xz^2+ze^z,\ x,\ 3+x-z^3)$.

$e^{xy} > 0$, so

$z \le 0 \iff x^2+y^2 \le 1$.

So $\partial S = C$ as drawn.



$\nabla \cdot \vec{F} = 3z^2 + 0 - 3z^2 = 0 \quad \Rightarrow \vec{F}$ is surf. ind.

And $D$ as drawn, oriented downward, has $\partial D = C = \partial S$
and $\vec{n} = (0,0,-1)$. So

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F} \cdot d\vec{S} = \iint_D \vec{F} \cdot \vec{n}\, dS$$

$$= \iint_D -3 - x + z^3\, dS$$

$$= \underbrace{\iint_D (-3)\, dS}_{=\,-3(\text{area})} + \underbrace{\iint_D (-x)\, dS}_{\substack{=\,0 \text{ by symm.} \\ \text{through } yz\text{-plane}}} + \underbrace{\iint_D (z^3)\, dS}_{\substack{=\,0 \text{ b/c } z=0 \\ \text{on } D.}}$$

$$= -3\pi$$

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