# EXAM 2

Math 212, 2023 Summer Term 2, Clark Bray.

Name: **Solutions**          Section:_____     Student ID:_____

## GENERAL RULES

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING TO RECEIVE CREDIT.
CLARITY WILL BE CONSIDERED IN GRADING.

No notes, no books, no calculators.

All answers must be reasonably simplified.

All of the policies and guidelines on the class webpages are in effect on this exam.

## WRITING RULES

Do not remove the staple, tear pages out of the staple, or tamper with the exam packet in any way. Do not write anything near the staple – this may be cut off.

Use black pen only. You may use a pencil for initial sketches of diagrams, but the final sketch must be drawn over in black pen and you must wipe all erasure residue from the paper.

Work for a given question can be done ONLY on the front or back of the page the question is written on. Room for scratch work is available on the back of this cover page, and on the two blank pages at the end of this packet; scratch work will NOT be graded.

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## DUKE COMMUNITY STANDARD STATEMENT

"I have adhered to the Duke Community Standard in completing this examination."

Signature: _____

1. *(15 pts)* A polygonal field has vertices at $(0,0)$, $(-1,2)$, $(5,2)$, and $(3,0)$. Flowers are growing such that the number of flowers per unit area is given by $\delta(x,y) = 100 + x + y$. Compute the total number of flowers in the field.

$$F = \iint dF = \iint \delta \, dA$$

$$= \int_0^2 \int_{-y/2}^{y+3} 100 + x + y \, dx \, dy$$



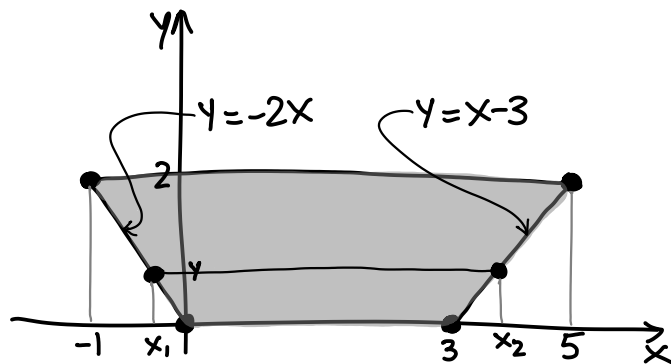$$= \int_0^2 \left( 100x + \tfrac{1}{2}x^2 + xy \right]_{x=-y/2}^{x=y+3} \, dy$$

$$= \int_0^2 \left( \left(100(y+3) + \tfrac{1}{2}(y+3)^2 + (y+3)y\right) - \left(100\left(\tfrac{-y}{2}\right) + \tfrac{1}{2}\left(\tfrac{-y}{2}\right)^2 + \left(\tfrac{-y}{2}\right)y\right) \right) dy$$

$$= \int_0^2 \left( \tfrac{15}{8} y^2 + 156y + \tfrac{609}{2} \right) dy$$

$$= \tfrac{15}{24} y^3 + 78 y^2 + \tfrac{609}{2} y \Big]_0^2$$
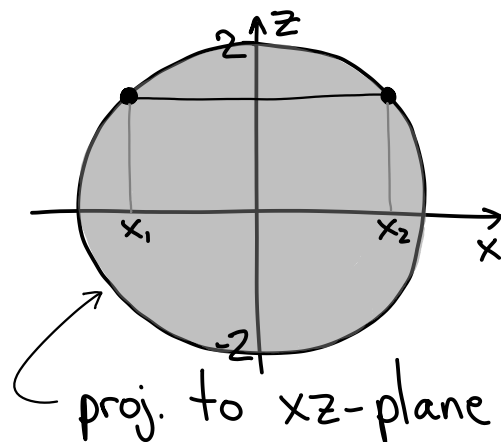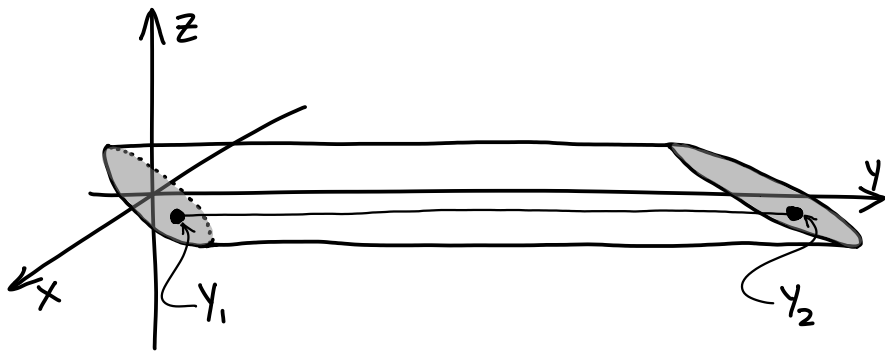
$$= 5 \qquad + 312 \qquad + 609$$

$$= 926$$

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2. *(20 pts)* $D$ is the solid region bounded by the surfaces with equations $x^2+z^2 = 4$, $x+y+2z = 0$, and $3x - y - z = 20$. Set up (but do not evaluate) a triple iterated integral representing $\iiint_D e^z \, dV$.



proj. to xz-plane

$$\iiint_D e^z \, dv \;=\; \int_{-2}^{2} \int_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}} \int_{-x-2z}^{3x-z-20} e^z \, dy \, dx \, dz$$

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3. *(30 pts)* The region $R$ in the $xy$-plane is bounded by the ellipse $(2x)^2 + (3y)^2 = 1$. We consider here the integral $\iint_R x^3 \, dx \, dy$.

(a) Use a change of variables to rewrite the above integral as an integral over the unit disk $D$ in the $uv$-plane with boundary equation $u^2 + v^2 = 1$. *(Hint: Let the equations suggest a change of variables function.)*

$$u = 2x \\ v = 3y \quad \Rightarrow \quad \frac{\partial(u,v)}{\partial(x,y)} = \det\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} = 6 \quad \Rightarrow \quad \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{6}$$

$$\iint_R x^3 \, dx \, dy = \iint_D \left(\frac{u}{2}\right)^3 \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \, du \, dv = \frac{1}{48} \iint_D u^3 \, du \, dv$$

(b) Compute the integral in the $uv$-plane resulting from (a) using polar coordinates.

$$\frac{1}{48} \iint_D u^3 \, du \, dv = \frac{1}{48} \int_0^{2\pi}\int_0^1 (r\cos\theta)^3 \, r \, dr \, d\theta = \frac{1}{48}\int_0^{2\pi} \left(\frac{1}{5}r^5\cos^3\theta\right]_{r=0}^{r=1} d\theta$$

$$= \frac{1}{240}\int_0^{2\pi}\cos^3\theta \, d\theta = \frac{1}{240}\int_0^{2\pi}\cos\theta - \sin^2\theta\cos\theta \, d\theta$$

$$= \frac{1}{240}\left(\sin\theta - \frac{1}{3}\sin^3\theta\right]_0^{2\pi} = 0 - 0 = 0$$

(c) Compute the original integral $\iint_R x^3 \, dx \, dy$ using the most efficient possible method (NOT the method from parts (a) and (b)!).

$R$ is symmetric over the $y$-axis, with reflection
$F(x,y) = (-x,y)$. $f(x,y) = x^3$ is odd because
$f(F(x,y)) = f(-x,y) = (-x)^3 = -x^3 = -f(x,y)$.
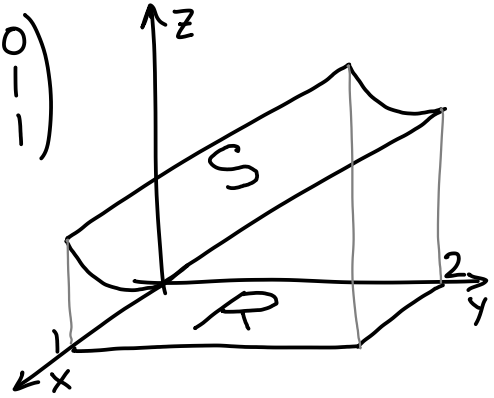So $\iint_R x^3 \, dx \, dy = 0$ by symmetry.

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4. *(15 pts)* $S$ is the portion of the graph of $f(x, y) = x^2 + y$ that is sitting over the rectangle $[0, 1] \times [0, 2]$ in the $xy$-plane. Set up, but do not evaluate, a double iterated integral to compute the surface area of $S$.

$$\vec{X} = \begin{pmatrix} u \\ v \\ u^2 + v \end{pmatrix} \implies \vec{X}_u = \begin{pmatrix} 1 \\ 0 \\ 2u \end{pmatrix}, \vec{X}_v = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$\implies \vec{N} = \begin{pmatrix} -2u \\ -1 \\ 1 \end{pmatrix} \implies \|\vec{N}\| = \sqrt{2 + 4u^2}$$



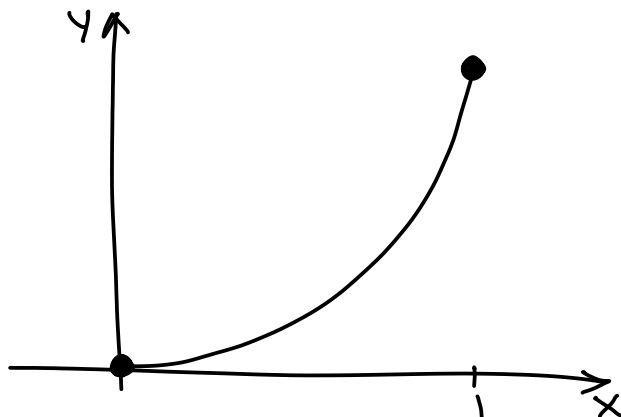$$area = \iint_R \|\vec{N}\| \, du\, dv = \int_0^2 \int_0^1 \sqrt{2 + 4u^2} \, du\, dv$$

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5. *(20 pts)* $C$ is the part of $y = x^2$ between $x = 0$ and $x = 1$, and $f(x, y) = x$. Compute $\int_C f(x, y) \, ds$.

$$\vec{X} = \begin{pmatrix} t \\ t^2 \end{pmatrix} \text{ on } t \in [0, 1]$$

$$\vec{X}' = \begin{pmatrix} 1 \\ 2t \end{pmatrix} \quad \|\vec{X}'\| = \sqrt{1 + 4t^2}$$

$$\int_C f(x, y) \, ds = \int_C x \, ds = \int_0^1 (t) \sqrt{1 + 4t^2} \, dt = \int_0^1 \sqrt{1 + 4t^2} \, (t \, dt)$$

$$\left( \text{Let } u = 1 + 4t^2, \text{ then } du = 8t \, dt. \right)$$

$$= \int u^{1/2} \left( \tfrac{1}{8} \, du \right) = \tfrac{1}{12} u^{3/2} \Big] = \tfrac{1}{12} \left( 1 + 4t^2 \right)^{3/2} \Big]_0^1 = \tfrac{1}{12} \left( 5^{3/2} - 1 \right)$$

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