EXAM 1

Math 212, 2021 Fall, Clark Bray.

Name: Solutions Section:______ Student ID:____________

GENERAL RULES

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

No notes, no books, no calculators.

All answers must be reasonably simplified.

All of the policies and guidelines on the class webpages are in effect on this exam.

WRITING RULES

Do not write anything near the staple – this will be cut off.

Use black pen only. You may use a pencil for initial sketches of diagrams, but the final sketch must be drawn over in black pen and you must wipe all erasure residue from the paper.

Work for a given question can be done ONLY on the front or back of the page the question is written on. Room for scratch work is available on the back of this cover page, and on the two blank pages at the end of this packet; scratch work will NOT be graded.

DUKE COMMUNITY STANDARD STATEMENT

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ________________________

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1. *(20 pts)* In this question we consider the points $\vec{p} = (1, 3, 2)$, $\vec{q} = (2, 1, 0)$, $\vec{r} = (1, 1, 1)$.

(a) Find the vector represented by the arrow whose tail is at $\vec{r}$, and whose head is at the midpoint of the line segment connecting $\vec{p}$ and $\vec{q}$.

$$\vec{m} = \frac{1}{2}(\vec{p}+\vec{q}) = \frac{1}{2}\begin{pmatrix} 1+2 \\ 3+1 \\ 2+0 \end{pmatrix} = \begin{pmatrix} 3/2 \\ 4/2 \\ 2/2 \end{pmatrix} = \begin{pmatrix} 3/2 \\ 2 \\ 1 \end{pmatrix}$$

$$\vec{v} = \vec{m} - \vec{r} = \begin{pmatrix} 3/2 - 1 \\ 2 - 1 \\ 1 - 1 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix}$$

(b) Compute the angle at $\vec{r}$ between the two segments connecting it to $\vec{p}$ and $\vec{q}$.

$$\vec{s}_1 = \vec{p} - \vec{r} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \qquad \vec{s}_2 = \vec{q} - \vec{r} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\theta = \arccos\left(\frac{\vec{s}_1 \cdot \vec{s}_2}{\|\vec{s}_1\| \, \|\vec{s}_2\|}\right) = \arccos\left(\frac{-1}{\sqrt{5}\sqrt{2}}\right) = \arccos\left(\frac{-1}{\sqrt{10}}\right)$$

(c) Compute the area of the parallelogram whose edge vectors are $\vec{p}$ and $\vec{q}$.

$$\vec{p} \times \vec{q} = \det\begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ 1 & 3 & 2 \\ 2 & 1 & 0 \end{pmatrix} = (-2, 4, -5)$$

$$\text{area} = \|\vec{p} \times \vec{q}\| = \sqrt{(-2)^2 + 4^2 + (-5)^2} = \sqrt{45}$$

(d) Make explicit use of some of your work from (c) to find the volume of the parallelepiped whose edge vectors are $\vec{p}$, $\vec{q}$, and $\vec{r}$.

$$\text{volume} = \left|\det\begin{pmatrix} \text{---}\ \vec{p}\ \text{---} \\ \text{---}\ \vec{q}\ \text{---} \\ \text{---}\ \vec{r}\ \text{---} \end{pmatrix}\right| = |\vec{p} \cdot (\vec{q} \times \vec{r})| = |\vec{r} \cdot (\vec{p} \times \vec{q})|$$

$$= |(1,1,1) \cdot (-2, 4, -5)| = |-3| = 3$$

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2. *(20 pts)* The parametric curve $\vec{x}(t) = (\sin t - \cos t + 2t, \sin t + 3\cos t, \sin t - 3\cos t - 3t)$ is entirely contained in a plane P. As a result, the velocity and acceleration vectors at every point on the curve must both be parallel to P. Use this fact to find the equation for the plane P.

$$\vec{v} = \vec{x}' = (\cos t + \sin t + 2,\ \cos t - 3\sin t,\ \cos t + 3\sin t - 3)$$

$$\vec{a} = \vec{v}' = (-\sin t + \cos t,\ -\sin t - 3\cos t,\ -\sin t + 3\cos t)$$

$$\Rightarrow \quad \vec{v}(0) = (3, 1, -2) \qquad \vec{a}(0) = (1, -3, 3)$$

These are both $\parallel$ to P so their cross product is $\perp$ to P.

$$\vec{n} = \vec{v}(0) \times \vec{a}(0) = \det\begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ 3 & 1 & -2 \\ 1 & -3 & 3 \end{pmatrix} = (-3, -11, -10)$$

Using $\vec{x}(0) = (-1, 3, -3)$ as $\vec{x}_0$ in the plane, the equation is

$$\vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{x}_0$$

$$-3x - 11y - 10z = 0$$

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3. *(20 pts)* The surface S with equation $x^2 + (3(y-2))^2 - z^2 + 1 = 0$ is the result of transforming the surface H with equation $x^2 + y^2 - z^2 + 1 = 0$ as follows:

(1) first, a translation by some amount (positive or negative) in the direction of one of the coordinate axes;

(2) then, a stretch by some positive factor (greater or less than 1) in the direction of one of the coordinate axes.

(a) What are the amount (positive or negative) and the coordinate axis referred to in (1)?

$$H: x^2+y^2-z^2+1=0 \xrightarrow{\text{"}y\text{"} \mapsto \text{"}y-6\text{"}} x^2+(y-6)^2-z^2+1=0$$

This is a translation by +6 in the y-direction.

(b) What are the factor (greater or less than 1) and the coordinate axis referred to in (2)?

$$x^2+(y-6)^2-z^2+1=0 \xrightarrow{\text{"}y\text{"} \mapsto \text{"}3y\text{"}} S: x^2+(3y-6)^2-z^2+1=0$$

This is a stretch by a factor of $\frac{1}{3}$ in the y-direction.

(c) Do either (or both) of S and H have any rotational symmetry? If so, identify the corresponding axes of symmetry and explain what indicates the symmetry.

S does not. H does, around the z-axis, because x and y appear only as part of "x^2+y^2".

(d) Is S best viewed as a graph of some function or as a level set of some function? Explain.

S cannot be viewed as a graph of any function!

It can be viewed as a level set, for example as the $g=0$ level set of $g: \mathbb{R}^3 \to \mathbb{R}^1$ defined by

$$g(x,y,z) = x^2+(3y-6)^2-z^2+1$$

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4. *(20 pts)* We know the following information about the graph $z = f(x,y)$ of the function $f : \mathbb{R}^2 \to \mathbb{R}^1$:

(a) it goes through the point $\vec{p} = (1,2,3)$; $\Rightarrow f(1,2) = 3$

(b) at $\vec{p}$, the cross section with the plane $x = 1$ has tangent line whose slope is 4; $\Rightarrow \frac{\partial f}{\partial y}(1,2) = 4$

(c) at $\vec{p}$, the cross section with the plane $y = 2$ has tangent line whose slope is 5. $\Rightarrow \frac{\partial f}{\partial x}(1,2) = 5$

Find the linear approximation of $f(1.06, 2.07)$.

choose $\vec{a} = (1,2)$, $d\vec{x} = (.06, .07)$

$\vec{x} = \vec{a} + d\vec{x} = (1.06, 2.07)$.

$$df = \nabla f(\vec{a}) \cdot d\vec{x}$$

$$= \begin{pmatrix} \frac{\partial f}{\partial x}(\vec{a}) \\ \frac{\partial f}{\partial y}(\vec{a}) \end{pmatrix} \cdot d\vec{x}$$

$$= \begin{pmatrix} 5 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} .06 \\ .07 \end{pmatrix}$$

$$= .58$$

$$f(\vec{x}) = f(\vec{a}) + df$$

$$= 3 + .58 = 3.58$$

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5. *(20 pts)* Suppose that $(x, y) = f(z, w)$ is differentiable, and we know specifically that when $(z, w) = (1, 2)$ we have $\frac{\partial x}{\partial w} = 1$, $\frac{\partial y}{\partial z} = 2$, $\frac{\partial y}{\partial w} = 3$, $\frac{\partial x}{\partial z} = 4$. If a particle $\vec{p}$ is at $(z, w) = (1, 2)$ and moving with velocity $(5, 6)$, what is the velocity of its image $f(\vec{p})$?

$$\frac{d\vec{p}}{dt} = \begin{pmatrix} 5 \\ 6 \end{pmatrix} \qquad J_f = \begin{pmatrix} \partial x/\partial z & \partial x/\partial w \\ \partial y/\partial z & \partial y/\partial w \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$$

J_f relates input velocities to output velocities, so

$$\frac{df}{dt} = J_f \frac{d\vec{p}}{dt}$$

$$= \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 26 \\ 28 \end{pmatrix}$$

Alt:

$$t \rightarrow \begin{matrix} z \\ w \end{matrix} \xrightarrow{f} \begin{matrix} x \\ y \end{matrix} \qquad \frac{dz}{dt} = 5, \quad \frac{dw}{dt} = 6$$

$$\frac{dx}{dt} = \frac{\partial x}{\partial z}\frac{dz}{dt} + \frac{\partial x}{\partial w}\frac{dw}{dt} = (4)(5) + (1)(6) = 26$$

$$\frac{dy}{dt} = \frac{\partial y}{\partial z}\frac{dz}{dt} + \frac{\partial y}{\partial w}\frac{dw}{dt} = (2)(5) + (3)(6) = 28$$

$$\frac{df}{dt} = \left(\frac{dx}{dt}, \frac{dy}{dt}\right) = (26, 28).$$

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