EXAM 3

Math 212, 2020 Fall.

Name: **Solutions**   NetID:_____   Student ID:_____

GENERAL RULES

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING TO RECEIVE CREDIT.
CLARITY WILL BE CONSIDERED IN GRADING.

No calculators.

All answers must be reasonably simplified.

All of the policies and guidelines on the class webpages are in effect on this exam.

It is strongly advised that you use black pen only, so that your work will scan as clearly as possible.

DUKE COMMUNITY STANDARD STATEMENT

"I have adhered to the Duke Community Standard in completing this examination."
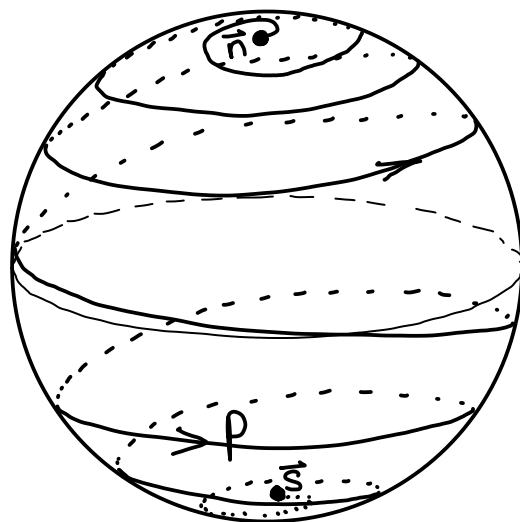
Signature: _____

1. *(16 pts)* The sphere M has radius 5 and center at the origin. The path P moves on M from the "north pole" to the "south pole" as described by $\theta = 10\phi$, for $0 \leq \phi \leq \pi$. Compute the line integral along P of the vector field

$$\vec{G} = \begin{pmatrix} y \\ x \\ z^2 \sin(z^3) \end{pmatrix}$$

$\nabla \times \vec{G} = \vec{0}$, so we know

$\vec{G} = \nabla f$ for some f.

$$f = \int y \, dx \qquad = xy \qquad + C_1(y,z)$$

$$= \int x \, dy \qquad = xy \qquad + C_2(y,z)$$

$$= \int z^2 \sin(z^3) \, dz = -\frac{1}{3}\cos(z^3) + C_3(x,y)$$

We can use $f = xy - \frac{1}{3}\cos(z^3)$.

Then

$$\int_P \vec{G} \cdot d\vec{x} = \int_P \nabla f \cdot d\vec{x} = f(\vec{s}) - f(\vec{n})$$

$$= \left(0 - \frac{1}{3}\cos(-125)\right) - \left(0 - \frac{1}{3}\cos(125)\right)$$

$$= 0 \qquad \left(\text{because } \cos \text{ is even} \atop \text{so } \cos(-125) = \cos(125).\right)$$

2. *(17 pts)* The curve E in the xy-plane is the counterclockwise-oriented ellipse with equation $(x-1)^2 + (2y-1)^2 = 8$.

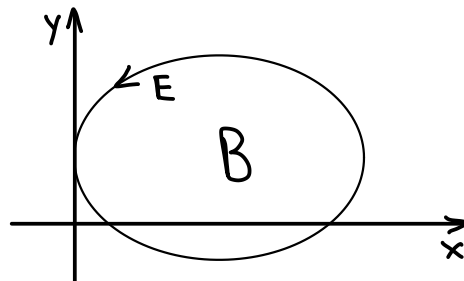   (a) Compute the circulation along E of the vector field

$$\vec{J} = \begin{pmatrix} xe^x - 4y \\ 3x - \ln(1+y^2) \end{pmatrix}$$

$$\int_E \vec{J} \cdot d\vec{x} = \iint_B \operatorname{grn} \vec{J} \; dA$$

$$= \iint_B 7 \, dA \;=\; 7 \cdot (\text{area of } B)$$

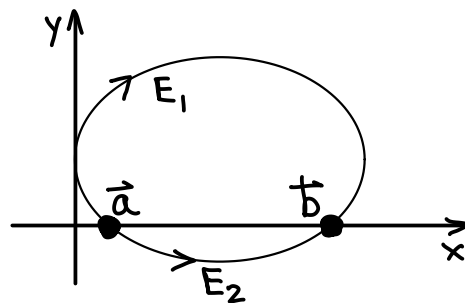$$= 7 \left(\pi (2\sqrt{2})(\sqrt{2}) \right)$$

$$= 28\pi$$

   (b) The curve E from part (a) intersects the x-axis at two points $\vec{a} = (a,0)$ and $\vec{b} = (b,0)$, with $a < b$. The paths E_1 and E_2 move along E from $\vec{a}$ to $\vec{b}$, above and below the x-axis respectively. What is the line integral of $\vec{J}$ along E_1 minus the the line integral of $\vec{J}$ along E_2?

Orientation on E_1 is opposite of E;
Orientation on E_2 is same as E.

So

$$\int_{E_1} \vec{J} \cdot d\vec{x} - \int_{E_2} \vec{J} \cdot d\vec{x} = - \int_E \vec{J} \cdot d\vec{x}$$

$$= -28\pi$$

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(extra space for questions from other side)

3. *(17 pts)* The surface S is the portion of the unit sphere with $z \geq 0$, oriented upward, and the surface D is the unit disk in the xy-plane, oriented upward. The vector field $\vec{F}$ is defined by
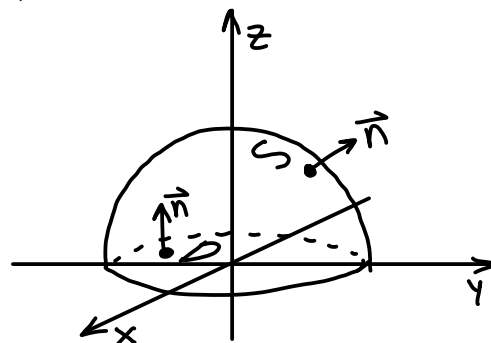
$$\vec{F}(x, y, z) = \begin{pmatrix} x^3 - 3x - 2 \\ ze^x - 4 \\ \sin y - 3x^2 z + 5 \end{pmatrix}$$

(a) Compute the flux of $\vec{F}$ through D.

$$\iint_D \vec{F} \cdot d\vec{S} = \iint_D \vec{F} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} dS$$

$$= \iint (\sin y - 3x^2 z + 5) \, dS$$

$$= \underbrace{\iint \sin y \, dS}_{\substack{=0 \text{ by symmetry} \\ \text{through } x\text{-axis.}}} - \underbrace{\iint 3x^2 z \, dS}_{\substack{=0 \text{ b/c } z=0 \\ \text{on } D.}} + 5\iint dS = 5\pi$$

(b) Use the result of part (a) to compute the flux of $\vec{F}$ through S.
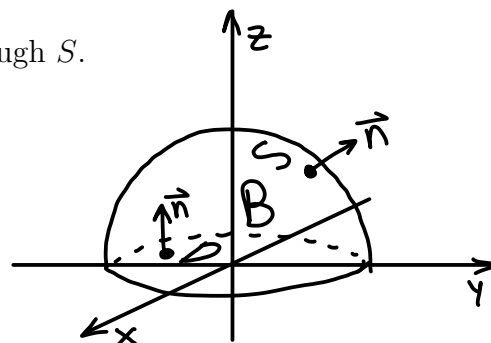
Let B be the solid enclosed by S and D, so $\partial B = S - D$.

$$\iint_S \vec{F} \cdot d\vec{S} - \iint_D \vec{F} \cdot d\vec{S} = \iint_{\partial B} \vec{F} \cdot d\vec{S}$$

$$= \iiint_B \nabla \cdot \vec{F} \, dV = \iiint_B (-3) \, dV = -2\pi$$

So

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F} \cdot d\vec{S} - 2\pi = 3\pi$$

4. *(17 pts)* A simple closed curve is a closed curve which does not intersect itself. The vector field $\vec{F}$ is continuously differentiable (all of its components are continuously differentiable), and has the following features:

  i. In every plane parallel to the xy-plane, along every simple closed curve oriented counterclockwise as seen from the positive direction of the z-axis, the circulation of $\vec{F}$ is 3 times the area enclosed by the curve;

  ii. In every plane parallel to the yz-plane, along every simple closed curve oriented counterclockwise as seen from the positive direction of the x-axis, the circulation of $\vec{F}$ is 5 times the area enclosed by the curve;

  iii. In every plane parallel to the xz-plane, along every simple closed curve oriented counterclockwise as seen from the positive direction of the y-axis, the circulation of $\vec{F}$ is 7 times the area enclosed by the curve.

Compute the line integrals of $\vec{F}$ along each of the curves indicated in parts (a) and (b) below.

Hints:

  1: *How might you rewrite the information given in i, ii, iii above?*

  2: *You may use the fact that, if k is a constant, $g : \mathbb{R}^2 \to \mathbb{R}^1$ is continuous, and $\iint_D g(x,y)\,dA = k$(the area of D) for all domains D, then $g(x,y) = k$.*

  3: *You are welcome to use parametrizations if they might be useful; but, each of the parts of this question below could be done without any parametrizing!*

(a) C moves through the plane $x + y + z = 5$ in straight line segments from $(5,0,0)$ to $(3,2,0)$ to $(0,0,5)$, and then back to $(5,0,0)$.

$$\text{Let } \nabla \times \vec{F} = \left(f_1, f_2, f_3 \right).$$

(i) tells us

$$3\left(\text{area of } D\right) = \int_{\partial D} \vec{F} \cdot d\vec{x} = \iint_D (\nabla \times \vec{F}) \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} dS = \iint_D f_3 \, dS$$

By Hint 2, we have $f_3 = 3$.

By similar arguments, (ii) & (iii) give us $f_1 = 5$ & $f_2 = 7$,

So $\nabla \times \vec{F} = (5, 7, 3)$.

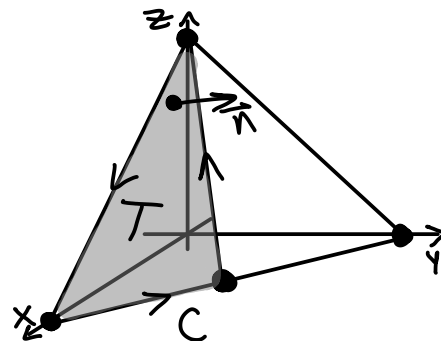$$\left(\text{cont'd}\right)$$

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Then

$$\int_C \vec{F} \cdot d\vec{x} = \iint_T (\nabla \times \vec{F}) \cdot \vec{n} \, dS$$

$$= \iint_T \begin{pmatrix} 5 \\ 7 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \frac{1}{\sqrt{3}} \, dS$$
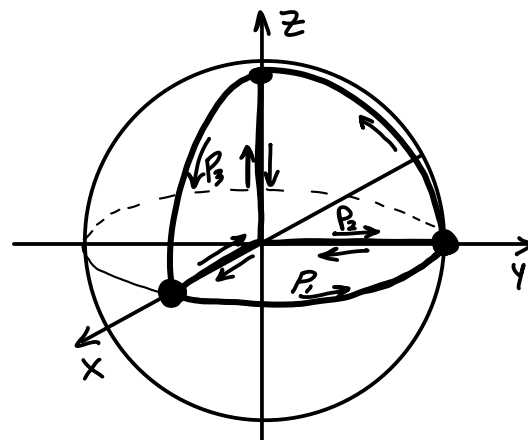
$$= \frac{15}{\sqrt{3}} (\text{area of } T)$$

$$= \frac{15}{\sqrt{3}} \left(\frac{1}{2} \left\| \begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix} \times \begin{pmatrix} -5 \\ 0 \\ 5 \end{pmatrix} \right\| \right) = \frac{15}{2\sqrt{3}} \left\| \begin{pmatrix} 10 \\ 10 \\ 10 \end{pmatrix} \right\| = 75$$

(b) P moves on the unit sphere in quarter-circle arcs, starting at $(1,0,0)$, then to $(0,1,0)$, then to $(0,0,1)$, and then back to $(1,0,0)$.

$P = P_1 + P_2 + P_3$ as indicated in the figure, due to orientation cancellations on the axes.
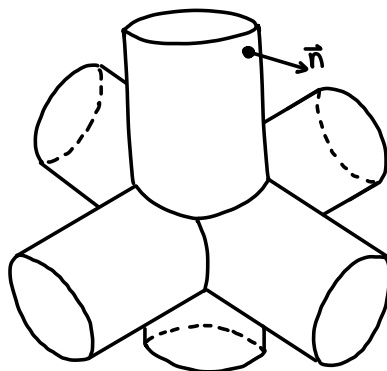
The line integrals on P_1, P_2, P_3 are addressed directly in i, ii, iii :

$$\int_{P_1} \vec{F} \cdot d\vec{x} = 3(\text{area of } P_1) = 3\left(\frac{\pi}{4}\right)$$

$$\int_{P_2} \vec{F} \cdot d\vec{x} = 5(\text{area of } P_1) = 5\left(\frac{\pi}{4}\right)$$

$$\int_{P_3} \vec{F} \cdot d\vec{x} = 7(\text{area of } P_1) = 7\left(\frac{\pi}{4}\right)$$

So $\int_P \vec{F} \cdot d\vec{x} = \frac{15\pi}{4}$

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5. *(17 pts)* The solid U is the union of $x^2 + y^2 \le 1$, $x^2 + z^2 \le 1$, $y^2 + z^2 \le 1$. The surface B is the boundary of U. The outward oriented surface S is the part of B inside the sphere of radius 2 centered at the origin. (See the figure above.)

Compute the flux through S of the vector field $\vec{F}$ below.

$$\nabla \cdot \vec{F} = 0, \text{ so}$$

$$\vec{F}(\vec{x}) = \begin{pmatrix} -yz^4 \\ xz^4 \\ 0 \end{pmatrix}$$
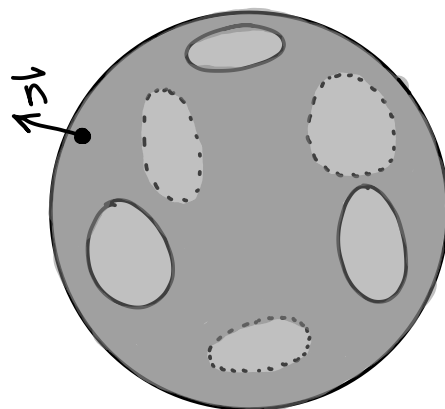
$\vec{F}$ is surface independent.

S has the same boundary as the portion P of the sphere with the "holes" removed.

So

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_P \vec{F} \cdot d\vec{S}$$

$$= \iint_P \begin{pmatrix} -yz^4 \\ xz^4 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tfrac{1}{2} \, dS$$

$$= \iint_P 0 \, dS = 0.$$

$$\llcorner \vec{n} = \frac{\vec{x}}{2}$$

6. *(16 pts)* Your friend wants to make a rectangular box, starting with a metal frame of 12 edges. The faces of the box will be made from a material that is free and plentiful, but each of the edges of the frame must be made from a particular expensive metal:

(a) the two vertical edges of the back face must be made of copper rod, which costs $1 per inch;

(b) the top and bottom edges of the back face must be made of pewter rod, which costs $2 per inch;

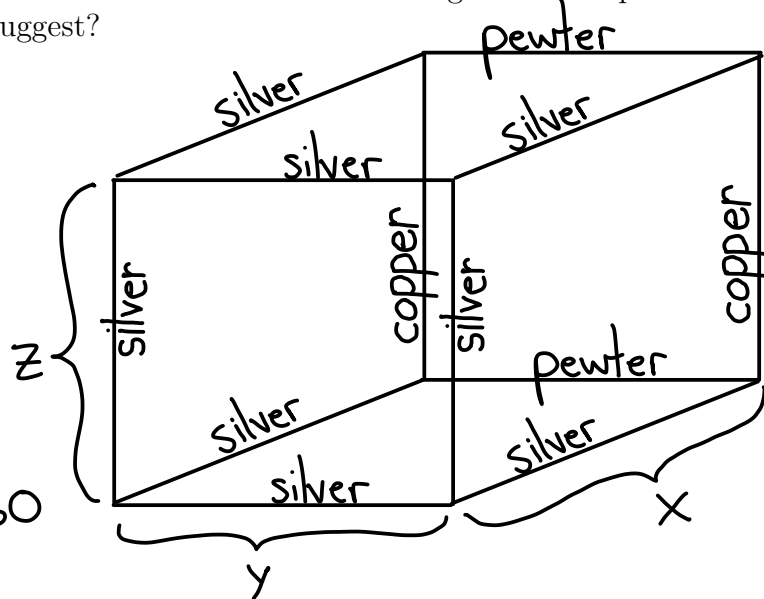(c) the other eight edges must be made of silver rod, which costs $3 per inch.

Your friend has $360 to spend on this box, and would like it to have the largest volume possible. What dimensions for the box should you suggest?

Want to maximize

$$f = xyz$$

Cost constraint is

$$g = 12x + 10y + 8z = 360$$

None of x, y, z can be negative; nor 0 because then the volume would be 0.

f is differentiable everywhere. $\nabla g = \begin{pmatrix} 12 \\ 10 \\ 8 \end{pmatrix}$ is never $\vec{0}$.

Lagrange condition is

$$\begin{pmatrix} yz \\ xz \\ xy \end{pmatrix} = \lambda \begin{pmatrix} 12 \\ 10 \\ 8 \end{pmatrix} \Rightarrow \frac{y}{12} = \frac{x}{10} \Rightarrow x = \frac{5}{6}y$$

$$\Rightarrow \frac{y}{8} = \frac{z}{10} \Rightarrow z = \frac{5}{4}y$$

Adding these to the constraint gives

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$$12\left(\frac{5}{6}y\right) + 10(y) + 8\left(\frac{5}{4}y\right) = 360$$

$$30y = 360$$

$$y = 12 \implies x = \frac{5}{6}y = 10$$

$$z = \frac{5}{4}y = 15$$