EXAM 1

Math 212, 2020 Fall.

Name: Solutions NetID:________ Student ID:________

GENERAL RULES

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

No calculators.

All answers must be reasonably simplified.

All of the policies and guidelines on the class webpages are in effect on this exam.

It is strongly advised that you use black pen only, so that your work will scan as clearly as possible.

DUKE COMMUNITY STANDARD STATEMENT

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ____________

(Scratch space. Nothing on this page will be graded!)

1. *(18 pts)* We consider the vectors $\vec{v} = (1, 3, 0)$, $\vec{w} = (0, 2, 1)$, $\vec{u} = (1, 0, 1)$.

(a) Find the vector $\vec{x} = k\vec{v}$ (where k is constant) for which $\vec{x} - \vec{w}$ is orthogonal to $\vec{w}$.

We need $(k\vec{v} - \vec{w}) \cdot \vec{w} = 0$

$$k(\vec{v} \cdot \vec{w}) - \|\vec{w}\|^2 = 0$$

$$k = \frac{\|\vec{w}\|^2}{\vec{v} \cdot \vec{w}} = \frac{5}{6}$$

So $\vec{x} = k\vec{v}$

$$= \frac{5}{6}\begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$$

(b) Compute the volume of the parallelepiped with edge vectors $\vec{v}$, $\vec{w}$, $\vec{u}$.

$$\det\begin{pmatrix} \text{---}\ \vec{v}\ \text{---} \\ \text{---}\ \vec{w}\ \text{---} \\ \text{---}\ \vec{u}\ \text{---} \end{pmatrix} = \det\begin{pmatrix} 1 & 3 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \end{pmatrix} = 1\,(2\cdot 1 - 1\cdot 0) - 3\,(0\cdot 1 - 1\cdot 1)$$

$$= 5$$

vol. $= |\det| = |5| = 5.$

(c) There are six ways to order the vectors $\vec{v}$, $\vec{w}$, $\vec{u}$. Identify which of these six are right-hand orders with a brief explanation. (For full credit you must do this with no redundant arithmetic.)

The determinant above shows that $\vec{v}, \vec{w}, \vec{u}$ is in RHO.

Cycling preserves order, so $\vec{u}, \vec{v}, \vec{w}$ and $\vec{w}, \vec{u}, \vec{v}$ are also RHO.

Transposing switches order, so $\vec{w}, \vec{v}, \vec{u}$ and $\vec{v}, \vec{u}, \vec{w}$ and $\vec{u}, \vec{w}, \vec{v}$ are in LHO.

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2. *(18 pts)* The curve C is entirely within a single plane P, parametrized with position given as $\vec{x}(t)$, and with velocity $\vec{v}(t)$ and acceleration $\vec{a}(t)$.

(a) Suppose we know only that $\vec{x}(0) = (2, -1, 0)$, $\vec{v}(0) = (-1, 0, 1)$, and $\vec{a}(0) = (-2, -1, 3)$. Find the equation of the plane P. *(Hint: What relationship must $\vec{v}$ and $\vec{a}$ have with P?)*

The motion is entirely in P, so $\vec{v}, \vec{a}$ must be parallel to P.

So $\vec{n} = \vec{v} \times \vec{a}$ is a normal vector.

$$\vec{n} = \vec{v} \times \vec{a} = \det\begin{pmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -1 & 0 & 1 \\ -2 & -1 & 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$\vec{x}_0 = \vec{x}(0)$ is given in the plane. So we have

$$(1,1,1)\cdot(x,y,z) = (1,1,1)\cdot(2,-1,0) \qquad \text{or} \qquad x+y+z = 1$$

(b) Suppose we know further that $\vec{v}(t) = (-2\sin t - \cos t, \cos t - e^t, e^t + 2\sin t)$. Find expressions for $\vec{x}(t)$ and $\vec{a}(t)$.

$$\vec{x} = \int \vec{v}\,dt + \vec{c} = \int \begin{pmatrix} -2\sin t - \cos t \\ \cos t - e^t \\ e^t + 2\sin t \end{pmatrix} dt + \vec{c} = \begin{pmatrix} 2\cos t - \sin t \\ \sin t - e^t \\ e^t - 2\cos t \end{pmatrix} + \vec{c}$$

$$t=0: \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} + \vec{c} \Rightarrow \vec{c} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Rightarrow x(t) = \begin{pmatrix} 2\cos t - \sin t \\ \sin t - e^t \\ e^t - 2\cos t + 1 \end{pmatrix}$$

$$\vec{a} = \vec{v}' = \begin{pmatrix} -2\cos t + \sin t \\ -\sin t - e^t \\ e^t + 2\cos t \end{pmatrix}$$

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3. *(16 pts)*

(a) The curve C is parametrized by $\vec{x}(t) = (1-t, 2t, 3t)$. Find the equation of the surface S obtained by rotating C around the z-axis. *(Hint: Find an expression for distance from a point on this parametrized curve to the z-axis, as a function of z.)*

Distance from $(1-t, 2t, 3t)$ to z-axis $(0,0,3t)$ is

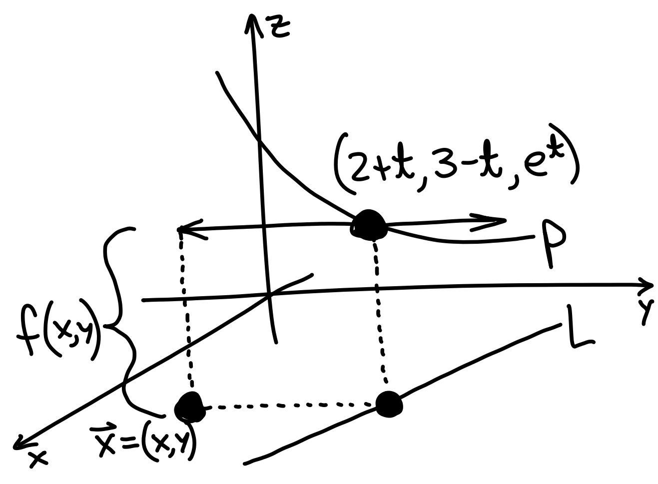
$$d = \sqrt{x^2+y^2} = \sqrt{(1-t)^2+(2t)^2} = \sqrt{5t^2-2t+1} = \sqrt{\tfrac{5}{9}z^2 - \tfrac{2}{3}z + 1}.$$

Rotating preserves this distance! We then write as

$$d^2 = \boxed{x^2+y^2 = \tfrac{5}{9}z^2 - \tfrac{2}{3}z + 1}$$

(b) The curve P is parametrized by $\vec{x}(t) = (2+t, 3-t, e^t)$. The surface R is formed by all of the translations of P in the y-direction (by vectors of the form $(0, k, 0)$). Find a function f whose graph is R.

The point on P whose height is f(x,y) is reached when it has the needed x-coordinate. So



$$x = 2 + t \Rightarrow t = x - 2$$

Then $z = e^t = e^{x-2}$. So $f(x,y) = e^{x-2}$.

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4. *(18 pts)* In this question we consider the function $f : \mathbb{R}^3 \to \mathbb{R}^1$ given by $f(x, y, z) = xy + e^{yz}$.

(a) Based on knowledge from the lectures, explain why f is continuously differentiable, noting all important details in the argument.

The partials are

$$\frac{\partial f}{\partial x} = y \qquad \frac{\partial f}{\partial y} = x + ze^{yz} \qquad \frac{\partial f}{\partial z} = ye^{yz}$$

These are all combinations of known continuous functions, so they are continuous.

Therefore f is continuously differentiable.

(b) Find a function h of the form $h(x, y, z) = c_1 + c_2 x + c_3 y + c_4 z$ that has the same value and partial derivatives as f at the point $(3, 1, 0)$.

$$f(3,1,0) = 4 \qquad \left.\frac{\partial f}{\partial x}\right|_{(3,1,0)} = 1 \qquad \left.\frac{\partial f}{\partial y}\right|_{(3,1,0)} = 3 \qquad \left.\frac{\partial f}{\partial z}\right|_{(3,1,0)} = 1$$

h is the linear approximation:

$$h(x,y,z) = f(3,1,0) + \left.\frac{\partial f}{\partial x}\right|_{(3,1,0)}(x-3) + \left.\frac{\partial f}{\partial y}\right|_{(3,1,0)}(y-1) + \left.\frac{\partial f}{\partial z}\right|_{(3,1,0)}(z-0)$$

$$= 4 + 1(x-3) + 3(y-1) + 1(z-0)$$

$$= -2 + x + 3y + z$$

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5. *(18 pts)*

(a) The function $p : \mathbb{R}^3 \to \mathbb{R}^3$ is given by $p(x, y, z) = (xe^y, y - z\cos x, xz - \sin z)$. A particle in the domain is moving as described by $\vec{x}(t) = (x(t), y(t), z(t)) = (1-t, 2+t, 3+4t)$. Find the velocity of the image $p(\vec{x}(t))$ at $t = 0$ WITHOUT finding an explicit expression for $p(\vec{x}(t))$.

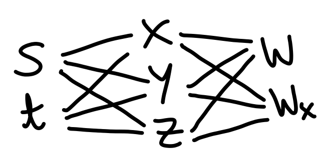
At $t=0$, $\vec{x}(t)$ has position $\vec{x}(0) = (1,2,3)$ and velocity $\vec{v} = (-1,1,4)$.

$$J_p = \begin{pmatrix} e^y & xe^y & 0 \\ z\sin x & 1 & -\cos x \\ z & 0 & x - \cos z \end{pmatrix} \qquad J_{p,\vec{x}(0)} = \begin{pmatrix} e^2 & e^2 & 0 \\ 3\sin 1 & 1 & -\cos 1 \\ 3 & 0 & 1-\cos 3 \end{pmatrix}$$

$$\text{velocity of } p(\vec{x}(t)) = \left(p(\vec{x}(t))\right)' = J_{p,\vec{x}(0)}\,\vec{x}\,'(t) = J_{p,\vec{x}(0)}\,\vec{v}$$

$$= \begin{pmatrix} e^2 & e^2 & 0 \\ 3\sin 1 & 1 & -\cos 1 \\ 3 & 0 & 1-\cos 3 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ -3\sin 1 + 1 - 4\cos 1 \\ 1 - 4\cos 3 \end{pmatrix}$$

(b) The variable w is a twice continuously differentiable function of x, y, and z. Also, x, y, and z (unrelated to part (a) above) are functions of s and t given as $x = s - t$, $y = 2s + t$, and $z = 3t - s$. Find a fully simplified expression for



s constant! $\quad \dfrac{\partial}{\partial t}(s\,y\,w_x)$

$$= s\left(\frac{\partial y}{\partial t} w_x + y\,\frac{\partial w_x}{\partial t}\right)$$

$$= s w_x + s y\left(\frac{\partial w_x}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w_x}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w_x}{\partial z}\frac{\partial z}{\partial t}\right)$$

$$= s w_x + s y\left(-w_{xx} + w_{xy} + 3 w_{xz}\right)$$

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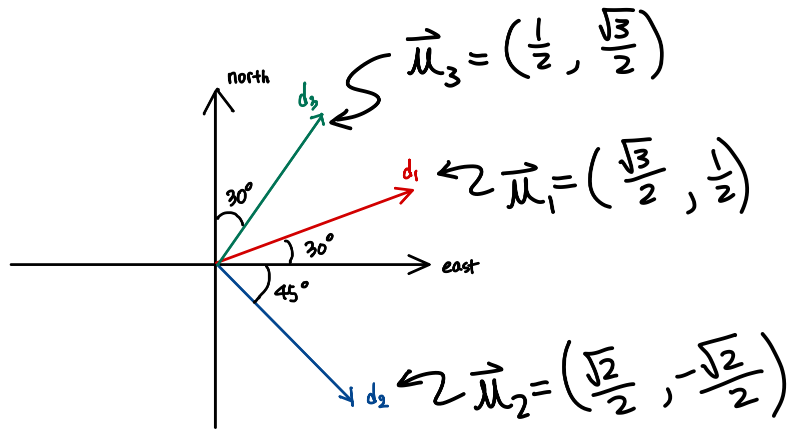

6. *(12 pts)* You are standing at a certain point $\vec{a}$ on a smooth hill. As you face in the direction d_1 (see the figure above) the slope in that direction is 0.3 uphill; as you face in the direction d_2 the slope in that direction is 0.2 downhill.

   Suppose that from this point you start walking in the direction d_3. How steep will your path on the hill be initially?

Let height be given by $f(x,y)$. Then we have

① $$0.3 = D_{\vec{u}_1} f(\vec{a}) = \nabla f(\vec{a}) \cdot \vec{u}_1 = \frac{\sqrt{3}}{2} f_x + \frac{1}{2} f_y$$

② $$-0.2 = D_{\vec{u}_2} f(\vec{a}) = \nabla f(\vec{a}) \cdot \vec{u}_2 = \frac{\sqrt{2}}{2} f_x - \frac{\sqrt{2}}{2} f_y$$

We can then solve for f_x, f_y:

$$\left.\begin{aligned} &① \quad 0.3 = \frac{\sqrt{3}}{2} f_x + \frac{1}{2} f_y \\ &② \overset{\cdot\frac{\sqrt{2}}{2}}{\Longrightarrow} \quad -\frac{\sqrt{2}}{10} = \frac{1}{2} f_x - \frac{1}{2} f_y \end{aligned}\right\} \Longrightarrow \frac{3-\sqrt{2}}{10} = \frac{1+\sqrt{3}}{2} f_x$$

$$\Rightarrow f_x = \frac{3-\sqrt{2}}{5(1+\sqrt{3})}$$

$$\left.\begin{aligned} &① \overset{\cdot\sqrt{2}}{\Longrightarrow} \quad \frac{3\sqrt{2}}{10} = \frac{\sqrt{6}}{2} f_x + \frac{\sqrt{2}}{2} f_y \\ &② \overset{\cdot\sqrt{3}}{\Longrightarrow} \quad \frac{-2\sqrt{3}}{10} = \frac{\sqrt{6}}{2} f_x - \frac{\sqrt{6}}{2} f_y \end{aligned}\right\} \Longrightarrow \frac{3\sqrt{2}+2\sqrt{3}}{10} = \frac{\sqrt{2}+\sqrt{6}}{2} f_y$$

$$\Rightarrow f_y = \frac{3\sqrt{2}+2\sqrt{3}}{5(\sqrt{2}+\sqrt{6})}$$

Then d_3-slope $= D_{\vec{u}_3} f(\vec{a}) = \nabla f(\vec{a}) \cdot \vec{u}_3$

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$$= \begin{pmatrix} \dfrac{3-\sqrt{2}}{5(1+\sqrt{3})} \\ \\ \dfrac{3\sqrt{2}+2\sqrt{3}}{5(\sqrt{2}+\sqrt{6})} \end{pmatrix} \cdot \begin{pmatrix} \dfrac{1}{2} \\ \\ \dfrac{\sqrt{3}}{2} \end{pmatrix} = \frac{3-\sqrt{2}}{10(1+\sqrt{3})} + \frac{3\sqrt{3}+3\sqrt{2}}{10(1+\sqrt{3})}$$

$$= \frac{3+3\sqrt{3}\ +2\sqrt{2}}{10(1+\sqrt{3})}$$