EXAM 3

Math 212, 2020 Spring, Clark Bray.

Name: Solutions Section:______ Student ID:____________

GENERAL RULES

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

No calculators.

All answers must be reasonably simplified.

All of the policies and guidelines on the class webpages are in effect on this exam.

WRITING RULES

Use black pen only. You may use a pencil for initial sketches of diagrams, but the final sketch must be drawn over in black pen and you must wipe all erasure residue from the paper.

DUKE COMMUNITY STANDARD STATEMENT

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ________________________

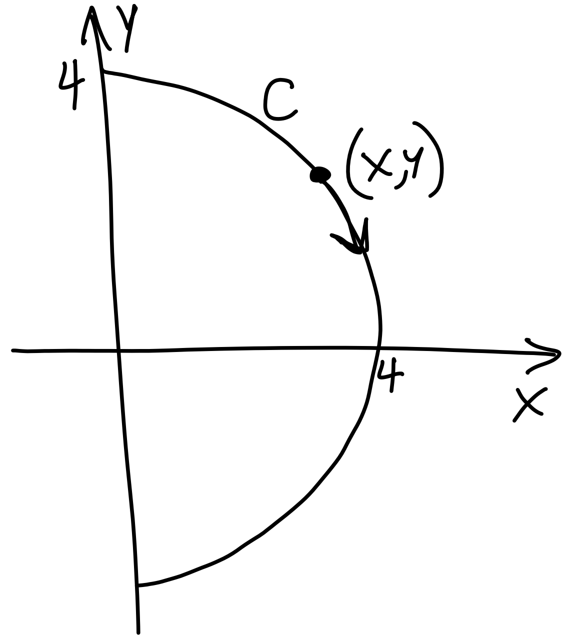
(Scratch space. Nothing on this page will be graded!)

1. *(20 pts)* As a function of location (x, y), the wind applies a force given by

$$\vec{F}(x, y) = (3 - x + 2y, 3x + y - 2)$$

The curve C is the circle of radius 4 centered at the origin. Compute the amount of work that is required to move clockwise along C from $(0, 4)$ to $(0, -4)$.

$$\text{grn}\,\vec{F} = Q_x - P_y = 3 - 2$$
$$= 1 \neq 0$$



So there is no antigradient.

C is parametrized backward by $\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4\cos t \\ 4\sin t \end{pmatrix}$

Then

$$W = \int_C (-\vec{F}) \cdot d\vec{x} = -\int_{-\pi/2}^{\pi/2} \left(-\vec{F}(\vec{x}(t))\right) \cdot \vec{x}\,'(t)\, dt$$

$$= \int_{-\pi/2}^{\pi/2} \begin{pmatrix} 3 - 4\cos t + 2(4\sin t) \\ 3(4\cos t) + 4\sin t - 2 \end{pmatrix} \cdot \begin{pmatrix} -4\sin t \\ 4\cos t \end{pmatrix} dt$$

(over)

(extra space for questions from other side)

$$= \int_{-\pi/2}^{\pi/2} \Big(-12\sin t + 32 \sin t \cos t - 32 \sin^2 t + 48 \cos^2 t - 8 \cos t \Big)\, dt$$

$$= \int_{-\pi/2}^{\pi/2} \Big(-12\sin t + 16 \sin 2t - 16(1 - \cos 2t) + 24(1 + \cos 2t) - 8 \cos t \Big)\, dt$$

$$= \int_{-\pi/2}^{\pi/2} 8 - 8\cos t \; dt$$

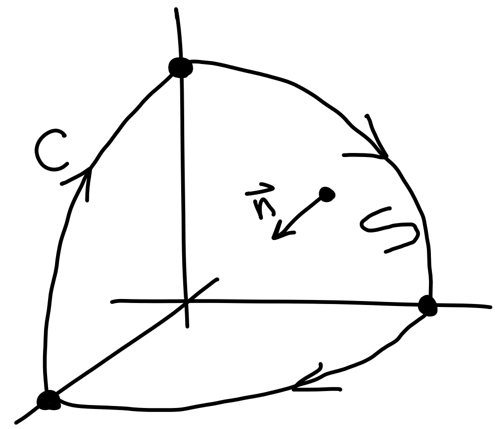
$$= 8t - 8\sin t \Big]_{-\pi/2}^{\pi/2} = 8\pi - 16$$

2. *(20 pts)* The curve P starts at $(1,0,0)$, then moves in quarter circle arcs to $(0,0,1)$, then $(0,1,0)$, then back to $(1,0,0)$, as parametrized by

$$\vec{x}(t) = \begin{cases} (\cos t, 0, \sin t) & 0 \le t \le \pi/2 \\ (0, -\cos t, \sin t) & \pi/2 \le t \le \pi \\ (-\sin t, -\cos t, 0) & \pi \le t \le 3\pi/2 \end{cases}$$

Compute the line integral along P of the vector field

$$\vec{F}(x,y,z) = \begin{pmatrix} x^2 e^x \\ 2 - \sin \pi y \\ x^2 + y^2 - z^2 \end{pmatrix}$$



$C = \partial S$, where S is the first octant portion of the unit sphere, oriented toward the origin.

$$\int_P \vec{F} \cdot d\vec{x} = \iint_S (\nabla \times \vec{F}) \cdot \vec{n} \, dS$$

$$= \iint_S \begin{pmatrix} 2y \\ -2x \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -x \\ -y \\ -z \end{pmatrix} dS$$

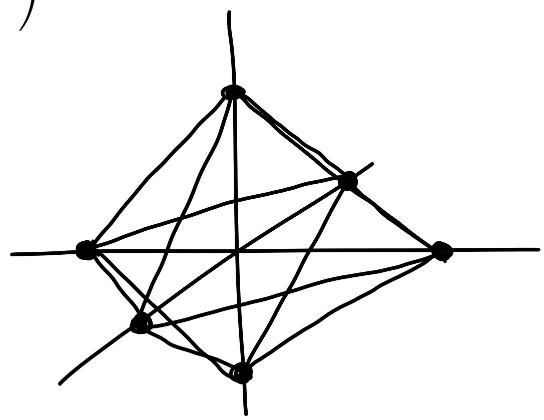
$$= \iint_S 0 \, dS = 0$$

(extra space for questions from other side)

3. *(20 pts)* The surface S is the octahedron with vertices at $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$, $(0, 0, \pm 1)$, oriented toward the origin. Compute the flux through S of the vector field

$$\vec{G}(x, y, z) = \begin{pmatrix} 3xy + e^y - 1 \\ xe^x - 3z^3 \\ x^2 y - 3z \end{pmatrix}$$

S is the boundary of the solid octahedron R, with the opposite orientation



$$\iint_S \vec{G} \cdot d\vec{S} = -\iiint_R \nabla \cdot \vec{G} \, dV$$

$$= -\iiint_R (3y - 3) \, dV$$

$$= \underbrace{\iiint_R (-3y) \, dV}_{\text{this is 0 by symmetry through the xz-plane}} + \iiint_R 3 \, dV$$

$$= 3(\text{volume of } R)$$

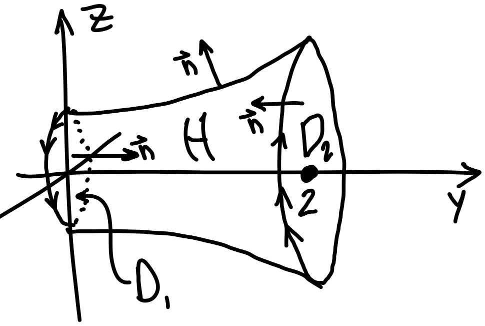
$$= 24(\text{volume of tetrahedron})$$

$$= 24\left(\frac{1}{6}\right) = 4$$

(extra space for questions from other side)

4. *(20 pts)* The surface H is the part of $x^2 + z^2 = e^y$ with $0 \le y \le 2$, oriented away from the y-axis. Compute the flux through H of the vector field

$$\vec{M}(x, y, z) = \begin{pmatrix} z - 2xy \\ y^2 + 2 \\ x^2 + y^2 \end{pmatrix}$$



$$\nabla \cdot \vec{M} = (-2y) + (2y) + (0)$$
$$= 0$$

So $\vec{M}$ is surface independent,

Then

$$\iint_H \vec{M} \cdot d\vec{S} = \iint_{D_1} \vec{M} \cdot d\vec{S} + \iint_{D_2} \vec{M} \cdot d\vec{S}$$

$$= \iint_{D_1} \underbrace{\begin{pmatrix} \cdot \\ y^2+2 \\ \cdot \end{pmatrix}}_{y=0} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} dS + \iint_{D_2} \underbrace{\begin{pmatrix} \cdot \\ y^2+2 \\ \cdot \end{pmatrix}}_{y=2} \cdot \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} dS$$

$$= 2(\text{area of } D_1) - 6(\text{area of } D_2)$$

$$= 2(\pi) - 6(\pi e^2)$$

$$= \pi(2 - 6e^2)$$

(extra space for questions from other side)

5. *(20 pts)* Find the "highest point" (greatest value of z) on the rotated ellipsoid with equation $3x^2 - 2xy + y^2 + 4z^2 + xz = 3$.

Want to maximize $f(\vec{x}) = z$ with constraint

$g = 3$, and $g(x,y,z) = 3x^2 - 2xy + y^2 + 4z^2 + xz$.

$$\nabla f = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \nabla g = \begin{pmatrix} 6x - 2y + z \\ -2x + 2y \\ x \quad + 8z \end{pmatrix}$$

f is differentiable, giving no critical points.

$\nabla g = \vec{0}$ only at $\vec{x} = \vec{0}$ not on ellipsoid, so no degenerate critical points.

Lagrange critical points:

$$\nabla f = \lambda \nabla g$$

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \lambda \begin{pmatrix} 6x - 2y + z \\ -2x + 2y \\ x \quad + 8z \end{pmatrix} \Rightarrow y = x \qquad z = -4x$$

The constraint equation then becomes

$$3x^2 - 2xy + y^2 + 4z^2 + xz = 3$$

$$3x^2 - 2x^2 + x^2 + 64x^2 - 4x^2 = 3$$

$$62x^2 = 3$$

(over)

(extra space for questions from other side)

$$x = \pm\sqrt{\frac{3}{62}}$$

$$\Rightarrow \vec{x} = \begin{pmatrix} \sqrt{3/62} \\ \sqrt{3/62} \\ -4\sqrt{3/62} \end{pmatrix} \text{ or } \begin{pmatrix} -\sqrt{3/62} \\ -\sqrt{3/62} \\ 4\sqrt{3/62} \end{pmatrix}$$

Of these, this is the one with the greatest value of z.

(Scratch space. Nothing on this page will be graded!)

(Scratch space. Nothing on this page will be graded!)

(Scratch space. Nothing on this page will be graded!)

(Scratch space. Nothing on this page will be graded!)