## EXAM 2

Math 212, 2019 Fall, Clark Bray.


## GENERAL RULES

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

No notes, no books, no calculators.
All answers must be reasonably simplified.
All of the policies and guidelines on the class webpages are in effect on this exam.

## WRITING RULES

Do not write anything near the staple - this will be cut off.
Use black pen only. You may use a pencil for initial sketches of diagrams, but the final sketch must be drawn over in black pen and you must wipe all erasure residue from the paper.

Work for a given question can be done ONLY on the front or back of the page the question is written on. Room for scratch work is available on the back of this cover page, and on the two blank pages at the end of this packet; scratch work will NOT be graded.

## DUKE COMMUNITY STANDARD STATEMENT

"I have adhered to the Duke Community Standard in completing this examination."

Signature: $\qquad$
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1. (20 pts) The domain $D$ is bounded by the coordinate axes in the $x y$-plane and the line $3 y-x=6$. Mass is distributed across $D$ with density $\delta(x, y)=y^{2}$. Compute the total mass in $D$.

$$
\begin{aligned}
& y \in[0,2] \\
& x \in[3 y-6,0]
\end{aligned}
$$



$$
\begin{aligned}
m & =\iint_{D} y^{2} d A \\
& =\int_{0}^{2} \int_{3 y-6}^{0} y^{2} d x d y \\
& =\int_{0}^{2} y^{2}(0-(3 y-6)) d y \\
& =\int_{0}^{2} 6 y^{2}-3 y^{3} d y \\
& \left.=2 y^{3}-\frac{3}{4} y^{4}\right]_{0}^{2} \\
& =4
\end{aligned}
$$

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2. (20 pts) The solid $R$ is bounded by the surfaces $y=0, y=z, y+z=4, x+2 y+3 z=20$, and $x=-4 z^{2}$. Mass is distributed across $R$ with density $\delta(x, y)=3$.
Write as an iterated triple integral (but do not evaluate it) the moment of inertia of $R$ around the line defined by $x=-1$ and $z=-2$.

$$
\begin{aligned}
I & =\iint_{R} r^{2} d m=\iint_{R} r^{2} \delta d V \\
& =3 \iint_{R}(x+1)^{2}+(z+2)^{2} d V
\end{aligned}
$$




$$
=3 \int_{0}^{2} \int_{y}^{4-y} \int_{-4 z^{2}}^{20-2 y-3 z}(x+1)^{2}+(z+2)^{2} d x d z d y
$$

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3. (20 pts) The solid polygon $S$ in the $x y$-plane has vertices at $(1,2),(-2,1),(-1,-2),(2,-1)$. Compute the integral over $S$ of the function $f(x, y)=3 x+y-2$.



$$
\begin{gathered}
\frac{\partial(\mu, v)}{\partial(x, y)}=\operatorname{det}\left(\begin{array}{cc}
1 & -3 \\
3 & 1
\end{array}\right)=10 \\
\left.\iint_{S}(3 x+y-2) d x d y=\iint_{0}(v-2) \left\lvert\, \frac{\partial(x, y)}{\partial(u, v)}\right.\right) d u d u \\
=\iint_{0}(v-2)\left(\frac{1}{10}\right) d u d v=\frac{1}{10} \underbrace{\iint_{D} v d u d v}_{1}-\frac{1}{5} \iint_{0} 1 d u d v
\end{gathered}
$$

$$
=-\frac{1}{5}(\operatorname{area} \text { of } D)
$$

$D$ is symmetric over the $\mu$-axis.

$$
=-20
$$

$$
R(\mu, v)=(\mu,-v) \text {, so } f(\mu, v)=v
$$

is odd because

$$
\begin{aligned}
f(R(\mu, v))= & f(\mu,-v)=-v \\
& =-f(\mu, v)
\end{aligned}
$$

So this integral is zero.
(extra space for questions from other side)
4. (20 pts) The surface $S$ has equation $z-z^{2}=\sqrt{x^{2}+y^{2}}$. Mass is enclosed inside of $S$ with density given by $\delta(x, y, z)=\sqrt{x^{2}+y^{2}}$. Set up an iterated integral (but do not evaluate) representing the total mass enclosed inside of $S$.
$S$ is rotationally symmetric around the $z$-axis, $z(1-z)=r$ has parabolic cross sections in $\theta$-planes and $r=0$ when $z=0,1$.



$$
\begin{aligned}
m & =\iiint d V \\
& =\int_{0}^{1} \int_{0}^{2 \pi} \int_{0}^{z-z^{2}} r(r d r d \theta d z)
\end{aligned}
$$

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5. (20 pts) For this question we assume that the surface of the earth is a perfect sphere of radius $R$. There is a latitude line $\alpha$ with the feature that the area between the equator and $\alpha$ is the same as the area north of $\alpha$. Find $\alpha$.
If it comes up in your calculations, you may use that

$$
\left\|\left(\begin{array}{c}
-\sin b \sin a \\
\sin b \cos a \\
0
\end{array}\right) \times\left(\begin{array}{c}
\cos b \cos a \\
\cos b \sin a \\
-\sin b
\end{array}\right)\right\|=|\sin b|
$$

We need the area north of $\alpha$ to be $1 / 4$ of the area $4 \pi R^{2}$ of the sphere.

Parametrize to compute $\|\vec{N}\|$ :

$$
\vec{X}=\left(\begin{array}{l}
R \sin \phi \cos \theta \\
R \sin \phi \sin \theta \\
R \cos \phi
\end{array}\right) \Rightarrow \vec{X}_{\phi}\left(\begin{array}{c}
R \cos \phi \cos \theta \\
R \cos \phi \sin \theta \\
-R \sin \phi
\end{array}\right), \vec{x}_{\theta}=\left(\begin{array}{c}
-R \sin \phi \sin \theta \\
R \sin \phi \cos \theta \\
0
\end{array}\right)
$$

Using the given algebra with $a=\theta, b=\phi$, we get

$$
\|\vec{N}\|=\left\|\vec{x}_{\phi} \times \vec{x}_{\theta}\right\|=R^{2}|\sin \phi|=R^{2} \sin \phi .
$$

Then the area north of $\alpha$ is

$$
\begin{aligned}
\pi R^{2} & =\int_{0}^{2 \pi} \int_{0}^{\frac{\pi}{2}-\alpha} R^{2} \sin \phi d \phi d \theta \\
& =\int_{0}^{2 \pi}\left(-R^{2} \cos \phi\right]_{0}^{\frac{\pi}{2}-\alpha} d \theta \\
& =\int_{0}^{2 \pi} R^{2}\left(1-\cos \left(\frac{\pi}{2}-\alpha\right)\right) d \theta
\end{aligned}
$$

(extra space for questions from other side)

$$
\begin{aligned}
& \pi R^{2}= 2 \pi R^{2}\left(1-\cos \left(\frac{\pi}{2}-\alpha\right)\right) \\
& \text { So } \cos \left(\frac{\pi}{2}-\alpha\right)=\frac{1}{2} \\
& \Rightarrow \frac{\pi}{2}-\alpha=\frac{\pi}{3} \\
& \Rightarrow \alpha=\frac{\pi}{6} \quad\left(30^{\circ} \text { north latitude }\right)
\end{aligned}
$$

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