## EXAM 3

Math 212, 2019 Spring, Clark Bray.
Name: Solutions
Section:_ Student ID:

## GENERAL RULES

## YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

No notes, no books, no calculators.
All answers must be reasonably simplified.
All of the policies and guidelines on the class webpages are in effect on this exam.

## WRITING RULES

Do not write anything on the QR codes or nearby print, or near the staple.
Use black pen only. You may use a pencil for initial sketches of diagrams, but the final sketch must be drawn over in black pen and you must wipe all erasure residue from the paper.

Work for a given question can be done ONLY on the front or back of the page the question is written on. Room for scratch work is available on the back of this cover page, and on the two blank pages at the end of this packet; scratch work will NOT be graded.

## DUKE COMMUNITY STANDARD STATEMENT

"I have adhered to the Duke Community Standard in completing this examination."
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1. (20 pts) The curve $C$ is parametrized by $\vec{x}(t)=(t / \pi, \cos t, \sin t)$, with $t \in[0,2 \pi]$. Compute the line integral over $C$ of the vector field $\vec{F}(x, y, z)=\left(y e^{x y}, 2 y z+x e^{x y}, y^{2}\right)$.
$\nabla \times \vec{F}=\overrightarrow{0}$, so we know there is an antigradient.

$$
\begin{aligned}
& f=\int y e^{x y} d x=e^{x y}+c_{1}(y, z) \\
& f=\int 2 y z+x e^{x y} d y=y^{2} z+e^{x y}+c_{2}(x, z) \\
& f=\int y^{2} d z \quad=y^{2} z \quad+c_{3}(x, y)
\end{aligned}
$$

So we can use $f=y^{2} z+e^{x y}$ as the antigradient. Using FTLI then we get

$$
\begin{aligned}
\int_{c} \vec{F} \cdot d \vec{x} & =\int_{c} \nabla f \cdot d \vec{x}=f(\vec{x}(2 \pi))-f(\vec{x}(0)) \\
& =f(2,1,0)-f(0,1,0) \\
& =e^{2}-1
\end{aligned}
$$

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2. (20 pts) The plane $P$ has equation $2 x+3 y+2 z=6$, and the surface $M$ is the part of $P$ with $x, y$, and $z$ all $\geq 0$, oriented toward the origin. Compute the flux through $M$ of the vector field $\vec{F}(x, y, z)=(1,0,2)$.
$z=3-x-\frac{3}{2} y$

$$
\vec{x}=\left(\begin{array}{l}
x \\
y \\
z
\end{array}\right)=\left(\begin{array}{c}
u \\
v \\
3-\mu-\frac{3}{2} v
\end{array}\right)
$$

$$
\vec{x}_{\mu}=\left(\begin{array}{c}
1 \\
0 \\
-1
\end{array}\right) \quad \vec{x}_{v}=\left(\begin{array}{c}
0 \\
1 \\
-3 / 2
\end{array}\right)
$$

$$
\vec{N}=\vec{x}_{u} \times \vec{x}_{v}=\left(\begin{array}{c}
1 \\
3 / 2 \\
1
\end{array}\right)<\underset{x=u}{ }
$$

 pointing the

$$
\Phi=\iint_{M} \vec{F} \cdot d \vec{S}=-\iint_{T} \vec{F} \cdot \vec{N} d u d v
$$ wrong way, so

wefix with this ( - ).

$$
\begin{aligned}
& =-\iint_{T}\left(\begin{array}{l}
1 \\
0 \\
2
\end{array}\right) \cdot\left(\begin{array}{c}
1 \\
3 / 2 \\
1
\end{array}\right) d u d v=\iint_{T}-3 d u d v \\
& =-3(\text { area of } T) \\
& =-3\left(\frac{1}{2} \cdot 3 \cdot 2\right)=-9
\end{aligned}
$$

(extra space for question from other side)
3. (20 pts) $S$ is the 5 -faced prism surface with vertices at the points $(0,0,0),(0,2,0),(0,0,3),(5,0,0)$, $(5,2,0),(5,0,3)$, oriented such that at the point $(2,1,0)$ the orientation is upward. Compute the total flux through these 5 faces of the vector field $\vec{G}(x, y, z)=(7-x+2 y, 2+y-z, 9-x+8 z)$.
$S$ is the inward-oriented boundary of the solid prism $P$.


$$
\begin{aligned}
\Phi & =\iint_{S} \vec{G} \cdot d \vec{S}=-\iint_{\partial P} \vec{G} \cdot d \vec{S}=-\iiint_{P} \nabla \cdot \vec{G} d V \\
& =-\iint_{P} 8 d V=-8(v d \cdot \text { of } P) \\
& =-8\left(\frac{1}{2} \cdot 2 \cdot 3 \cdot 5\right) \\
& =-120
\end{aligned}
$$

(extra space for question from other side)
4. (20 pts) The curve $C$ in $\mathbb{R}^{3}$ is the polygon with vertices at $(0,0,0),(0,3,0),(5,3,0),(5,2,0)$, $(1,2,0),(1,1,0),(5,1,0),(5,0,0)$, and then back to $(0,0,0)$, oriented in the direction of this listing of the vertices. Compute the circulation along $C$ of the vector field $\vec{H}(x, y, z)=\left(x e^{x}-2 y, x-\sin y, 4 x+y-z^{2}-9\right)$.
$C$ is the boundary of the
surface $S$, oriented downward.


$$
\text { circulation }=\int_{c} \vec{H} \cdot d \vec{x}=\iint_{S}(b x \vec{H}) \cdot d \vec{S}=\iint_{S}(\vec{x} \overrightarrow{\vec{k}}) \vec{n} d S
$$

$$
=\iint_{S}\binom{1}{3} \cdot\binom{0}{0-1} d S=-3(\text { area of } S)=-33
$$

(extra space for question from other side)
5. (20 pts) The surface $E$ is the part of the (outward oriented) ellipsoid $3 x^{2}+3 y^{2}+z^{2}=16$ that is on or below the plane $z=2$. Compute the flux through $E$ of the vector field $\vec{W}(x, y, z)=(x+y z, x z-2 y, 1+z)$.

$$
\nabla \cdot \vec{W}=1-2+1=0
$$

So $\vec{W}$ is surface independent.
The disk $D$ has the same oriented boundary as $E$, so


$$
\iint_{E} \vec{W} \cdot \overrightarrow{d S}=\iint_{D} \vec{W} \cdot d \vec{S}=\iint_{D}\binom{\vdots}{1+z} \cdot\left(\begin{array}{c}
0 \\
0 \\
-1
\end{array}\right) d S
$$

On $D$ we know $z=2$, so the integral becomes

$$
=\iint_{D}-3 d S=-3(\text { area of } D)
$$

With $3 x^{2}+3 y^{2}+z^{2}=16$ and $z=2$, the equation of $\partial 0$ becomes $x^{2}+y^{2}=4$ so the area of $D$ is $4 \pi$.

So flux $=-12 \pi$.
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