

EXAM 1

Math 212, 2019 Spring, Clark Bray.

Name: Solutions Section: _____ Student ID: _____

GENERAL RULES

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING TO RECEIVE CREDIT.
CLARITY WILL BE CONSIDERED IN GRADING.

No notes, no books, no calculators.

All answers must be reasonably simplified.

All of the policies and guidelines on the class webpages are in effect on this exam.

WRITING RULES

Do not write anything on the QR codes or nearby print, or near the staple.

Use black pen only. You may use a pencil for initial sketches of diagrams, but the final sketch must be drawn over in black pen and you must wipe all erasure residue from the paper.

Work for a given question can be done ONLY on the front or back of the page the question is written on. Room for scratch work is available on the back of this cover page, and on the two blank pages at the end of this packet; scratch work will NOT be graded.

DUKE COMMUNITY STANDARD STATEMENT

“I have adhered to the Duke Community Standard in completing this examination.”

Signature: _____

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1. (20 pts) Let $\vec{v} = (1, 3, 2)$, $\vec{w} = (0, 1, 1)$, $\vec{x} = (2, 1, 0)$.

(a) Compute the amount of work that it takes to displace by \vec{v} while applying a force $\vec{F} = \vec{w}$.

$$W = \vec{F} \cdot \vec{x} = \vec{w} \cdot \vec{v} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = 5$$

(b) Find the area of the parallelogram determined by \vec{w} and \vec{x} .

$$\begin{aligned} \text{area} &= \left\| \vec{w} \times \vec{x} \right\| = \left\| \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \right\| = \left\| \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix} \right\| \\ &= \sqrt{(-1)^2 + (2)^2 + (-2)^2} = 3 \end{aligned}$$

(c) Find the volume of the parallelepiped determined by \vec{v} , \vec{w} and \vec{x} .

$$\text{volume} = \left| \det(\vec{v}, \vec{w}, \vec{x}) \right| = \left| \vec{v} \cdot (\vec{w} \times \vec{x}) \right| = \left| \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix} \right| = |1| = 1$$

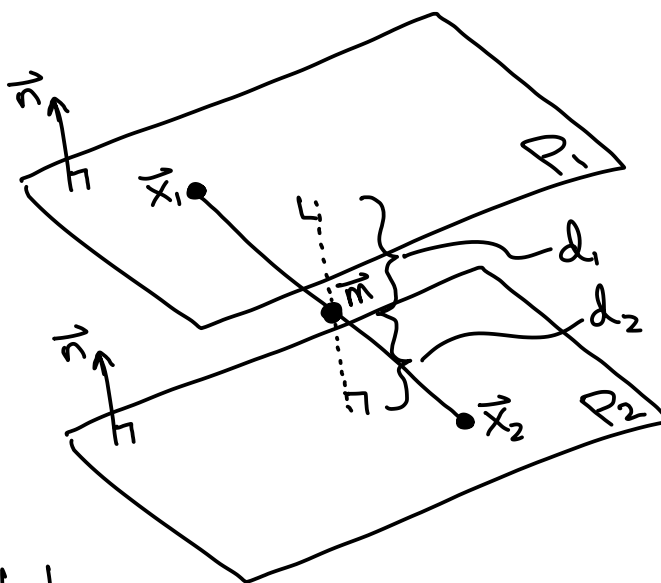
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2. (20 pts) The planes P_1 and P_2 have equations $\vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{x}_1$ and $\vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{x}_2$, respectively. Show that the point

$$\vec{m} = \frac{\vec{x}_1 + \vec{x}_2}{2}$$

is equidistant to P_1 and P_2 .

Planes have the same normal vector \vec{n} , so they are parallel as drawn.



We see from the figure that

$$d_1 = |\text{comp}_{\vec{n}}(\vec{x}_1 - \vec{m})|, \quad d_2 = |\text{comp}_{\vec{n}}(\vec{m} - \vec{x}_2)|$$

and algebra gives us

$$\begin{aligned} \vec{x}_1 - \vec{m} &= \vec{x}_1 - \frac{\vec{x}_1 + \vec{x}_2}{2} \\ &= \frac{\vec{x}_1 - \vec{x}_2}{2} \end{aligned}$$

$$\begin{aligned} \vec{m} - \vec{x}_2 &= \frac{\vec{x}_1 + \vec{x}_2}{2} - \vec{x}_2 \\ &= \frac{\vec{x}_1 - \vec{x}_2}{2} \end{aligned}$$

So d_1, d_2 are computed by identical expressions and are therefore equal.

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3. (20 pts) The curve C in \mathbb{R}^2 is parametrized by $(x, y) = (8t^2, 2t + 1)$.

(a) Find the equation of C , and identify what kind of curve it is.

$$\begin{aligned} y &= 2t + 1 \\ y^2 &= 4t^2 + 4t + 1 \\ 2y^2 &= \underbrace{8t^2}_x + \underbrace{8t + 2}_{4y - 2} \end{aligned} \quad \begin{array}{l} \text{So} \\ x = 2y^2 - 4y + 2 \\ \text{This is a parabola.} \end{array}$$

(b) For what value of t is this parametrization moving the slowest?

$$\begin{aligned} \vec{v}(t) &= \vec{x}'(t) = (16t, 2) \\ \text{speed} = v &= \|\vec{v}\| = \sqrt{256t^2 + 4} \end{aligned}$$

This is least when $t=0$.

(c) Compute the curvature of C at the point $(8, 3)$.

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 8t^2 \\ 2t+1 \end{pmatrix} = \begin{pmatrix} 8 \\ 3 \end{pmatrix} \Rightarrow 2t+1=3 \Rightarrow t=1$$

$$\vec{v}(t) = \begin{pmatrix} 16t \\ 2 \end{pmatrix}$$

$$\vec{a}(t) = \begin{pmatrix} 16 \\ 0 \end{pmatrix}$$

@ $t=1$ we have $\vec{v} = \begin{pmatrix} 16 \\ 2 \end{pmatrix}$
 $\vec{a} = \begin{pmatrix} 16 \\ 0 \end{pmatrix}$

$$\begin{aligned} \kappa &= \frac{|x'y'' - x''y'|}{v^3} \\ &= \frac{|16 \cdot 0 - 16 \cdot 2|}{(\sqrt{260})^3} \\ &= \frac{32}{(260)^{3/2}} \end{aligned}$$

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4. (20 pts) The surface S has equation $4x^2 + y + (z + 1)^2 = 0$.

- (a) Find the equation of a surface R , rotationally symmetric around one of the three coordinate axes, and a sequence of geometric transformations that could be applied to R to result in the above surface S .

rot. sym.
y-axis

squish by 2
in x-dir. with
" x " \mapsto " $2x$ "

translate by -1
in z-dir with
" z " \mapsto " $z+1$ "

$$\boxed{x^2 + y + z^2 = 0} \quad 4x^2 + y + z^2 = 0 \quad 4x^2 + y + (z+1)^2 = 0$$

- (b) Use the work above to identify S as either a sphere, ellipsoid, circular or elliptical paraboloid, circular or elliptical hyperboloid of one or two sheets, hyperbolic paraboloid, or cone.

R is a circular paraboloid. The squish in the x -direction makes an elliptical paraboloid (and the subsequent translation does not affect this).

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5. (20 pts) The curve K in \mathbb{R}^2 has equation $2x - 5y = 10$.

- (a) Is K the graph of a function p ? If so, identify the domain and target, and a formula for computing it; if not, explain why not.

The equation is equivalent to $y = \frac{2x-10}{5}$, so the curve is the graph $y = p(x)$ of $p: \mathbb{R}^1 \rightarrow \mathbb{R}^1$ defined by $p(x) = \frac{2x-10}{5}$.

- (b) Is K a level set of a function r ? If so, identify the domain and target, and a formula for computing it; if not, explain why not.

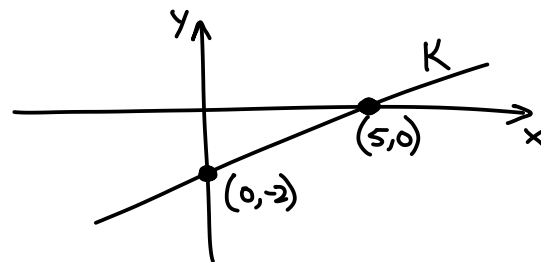
The equation $2x - 5y = 10$ shows that the curve is the level set $r = 10$ of the function $r: \mathbb{R}^2 \rightarrow \mathbb{R}^1$ defined by

$$r(x, y) = 2x - 5y$$

- (c) Is K parametrized by a function m ? If so, identify the domain and target, and a formula for computing it; if not, explain why not.

K is a line, and goes through $(5, 0)$ and $(0, -2)$, so it is parallel to $(5, 2)$. Then K is parametrized by $m: \mathbb{R}^1 \rightarrow \mathbb{R}^2$ defined by

$$m(t) = \begin{pmatrix} 0 \\ -2 \end{pmatrix} + t \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$



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