EXAM 3

Math 212, 2018 Fall, Clark Bray.

Name: Solutions Section:______ Student ID:____________

GENERAL RULES

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

No notes, no books, no calculators. Scratch paper is allowed, but (1) it must be from the instructor, (2) it must be returned with the exam, and (3) it will NOT be graded.

All answers must be reasonably simplified.

All of the policies and guidelines on the class webpages are in effect on this exam.

WRITING RULES

Do not write anything on the QR codes or near the staple.

Use black pen only. You may use a pencil for initial sketches of diagrams, but the final sketch must be drawn over in black pen and you must wipe all erasure residue from the paper.

Work for a given question can ONLY be done on the front or back of the page the question is written on.

DUKE COMMUNITY STANDARD STATEMENT

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ____________________

(Nothing on this page will be graded!)

1. *(20 pts)* Compute the line integral $\int_C \vec{F} \cdot d\vec{x}$, where C is parametrized by

$$\vec{x}(t) = (\cos^3 t, 1 + \cos t, \sin^3 t), \quad t \in [0, \pi]$$

and the vector field is given by $\vec{F}(x, y, z) = (y^2, 2xy - z^2, -2yz)$.

$$\nabla \times \vec{F} = \left(-2z - (-2z), 0 - 0, 2y - 2y\right) = \vec{0}$$

So there is an antigradient f.

$$\left.\begin{aligned} f &= \int y^2\,dx &&= xy^2 &&+ C_1(y,z) \\ f &= \int 2xy - z^2\,dy &&= xy^2 - yz^2 &&+ C_2(x,z) \\ f &= \int -2yz\,dz &&= \quad\ -yz^2 &&+ C_3(x,y) \end{aligned}\right\}$$

Can use $f = xy^2 - yz^2$

Then by the F.T.L.I. we have

$$\begin{aligned} \int_C \vec{F} \cdot d\vec{x} &= f(\vec{x}(\pi)) - f(\vec{x}(0)) \\ &= f(-1,0,0) - f(1,2,0) \\ &= 0 - (4) \\ &= -4 \end{aligned}$$

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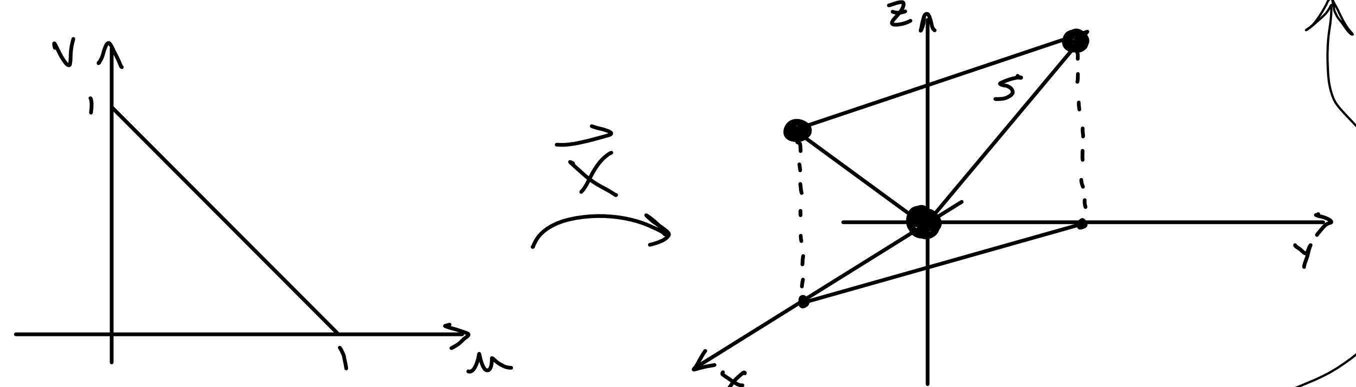
2. *(20 pts)* The surface S is the triangle with vertices at $(0,0,0)$, $(1,0,1)$, and $(0,1,1)$, oriented downward. Compute the flux through S of the vector field $\vec{G}(x,y,z) = (3,1,2-z)$.

The surface is not a boundary, so we cannot use Gauss's theorem.

Equation is $z = x+y$, so using the graph parametrization we get

$$\vec{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} u \\ v \\ u+v \end{pmatrix}, \quad \vec{X}_u = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad \vec{X}_v = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \vec{N} = \vec{X}_u \times \vec{X}_v = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$



Negative inserted due to parametrization being up, not down:

$$\Phi = -\iint \vec{F} \cdot d\vec{S} = -\iint \vec{F} \cdot \vec{N}\, du\, dv = -\iint \begin{pmatrix} 3 \\ 1 \\ 2-(u+v) \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} du\, dv$$

$$= -\int_0^1 \int_0^{1-v} (-2-u-v)\, du\, dv = -\int_0^1 \left(-2u - \frac{1}{2}u^2 - uv\right]_{u=0}^{u=1-v} dv$$

$$= -\int_0^1 -2(1-v) - \frac{1}{2}(1-v)^2 - (1-v)v\, dv$$

$$= -\int_0^1 \frac{1}{2}v^2 + 2v - \frac{5}{2}\, dv$$

$$= -\left(\frac{1}{6}v^3 + v^2 - \frac{5}{2}v\right]_0^1 = \boxed{\frac{4}{3}}$$

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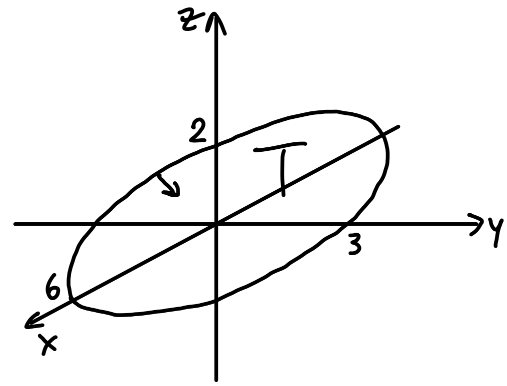
3. *(20 pts)* The ellipsoid E has equation $x^2 + 4y^2 + 9z^2 = 36$, and is oriented toward the origin. Compute the flux through E of the vector field $\vec{H}(x, y, z) = (2x + y - z, 3x - y + 4z, x + 5z)$.

E has opposite orientation of the boundary of the solid T.



So

$$\iint_E \vec{H} \cdot d\vec{S} = -\iint_{\partial T} \vec{H} \cdot d\vec{S} = -\iiint_T \nabla \cdot \vec{H} \, dV = -\iiint_T (6) \, dV$$

$$= -6 \left(\text{volume of } T \right) = -6 \left(\tfrac{4}{3}\pi (6 \cdot 3 \cdot 2) \right) = -288\pi$$

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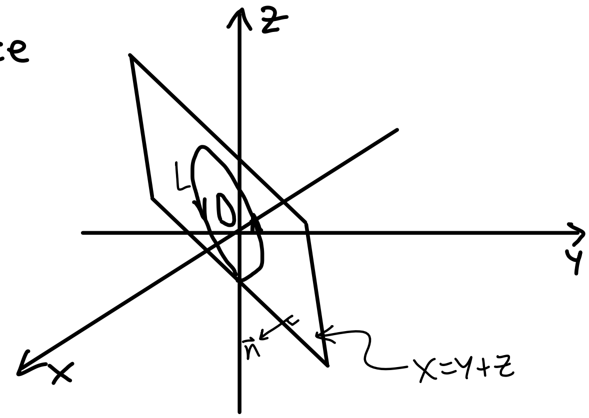
4. *(20 pts)* The loop L is in the plane $x = y+z$, and is parametrized by $\vec{x}(t) = (\cos t+\sin t, \cos t, \sin t)$. Compute the circulation around L of the vector field $\vec{J}(x,y,z) = (e^{x^2} + 2y, y^3 + 3z, x - 2z)$.

L is the boundary of the surface D in x=y+z.



$$\int_L \vec{J}\cdot d\vec{x} = \iint_D (\nabla\times\vec{J})\cdot d\vec{S}$$

$$= \iint_D \begin{pmatrix}-3\\-1\\-2\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\-1\end{pmatrix}\Big/\sqrt{3}\; dS$$

$$= 0$$

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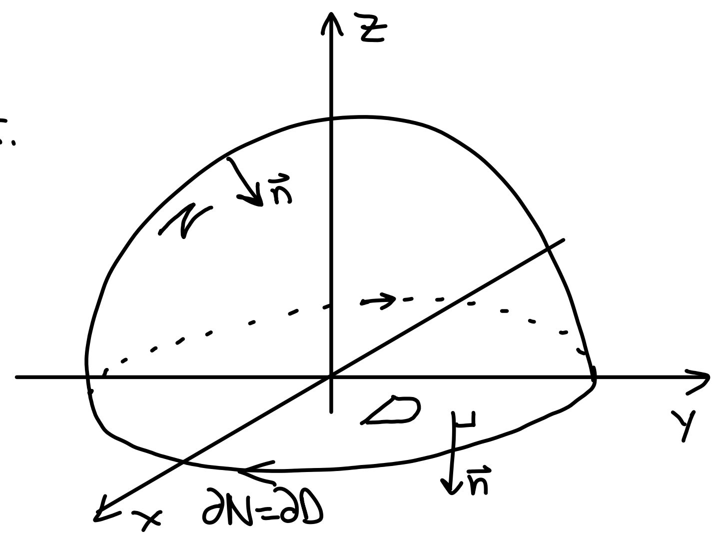
5. *(20 pts)* The surface N is the part of the unit sphere above the xy-plane, oriented inward. Compute the flux through N of the vector field $\vec{K}(x,y,z) = (x + y^3, 4z^2 - y, xy + 3)$.

$\nabla \cdot \vec{K} = 1 - 1 + 0 = 0.$

So K is surface independent.

D has the same boundary as N, including orientation.



So

$$\iint_N \vec{K} \cdot d\vec{S} = \iint_D \vec{K} \cdot d\vec{S}$$

$$= \iint_D \vec{K} \cdot \vec{n}\, dS$$

$$= \iint_D \vec{K} \cdot \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} dS$$

$$= \iint_D -(xy+3)\, dS$$

$$= \underbrace{-\iint_D xy\, dS}_{\substack{=0 \text{ by symmetry} \\ \text{over x-axis}}} - 3\underbrace{\iint_D 1\, dS}_{= \text{area} = \pi}$$

$$= -3\pi$$

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