EXAM 2

Math 212, 2018 Fall, Clark Bray.

Name: Solutions Section:______ Student ID:____________

GENERAL RULES

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

No notes, no books, no calculators. Scratch paper is allowed, but (1) it must be from the instructor, (2) it must be returned with the exam, and (3) it will NOT be graded.

All answers must be reasonably simplified.

All of the policies and guidelines on the class webpages are in effect on this exam.

WRITING RULES

Do not write anything on the QR codes or near the staple.

Use black pen only. You may use a pencil for initial sketches of diagrams, but the final sketch must be drawn over in black pen and you must wipe all erasure residue from the paper.

Work for a given question can ONLY be done on the front or back of the page the question is written on.

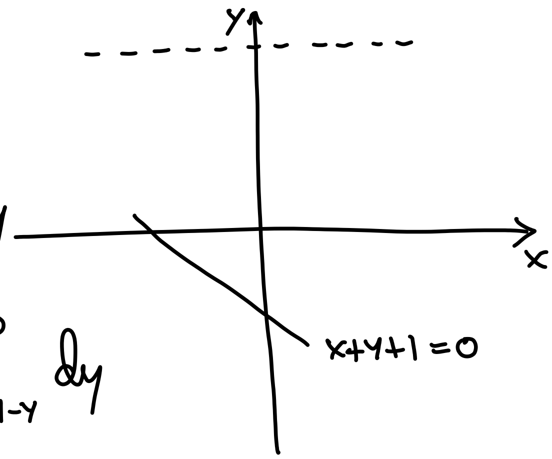
DUKE COMMUNITY STANDARD STATEMENT

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ________________________

(Nothing on this page will be graded!)

1. *(20 pts)* The region D in the xy-plane is bounded by the lines $x = 0$, $y = 0$, and $x + y + 1 = 0$. Mass is distributed over D with density given by $\delta(x, y) = -6x$. Compute the moment of inertia of this mass around the line with equation $y = 2$.



$$I = \iint_D r^2\, dm = \iint (2-y)^2\, \delta\, dA$$

$$= \int_{-1}^{0}\int_{-1-y}^{0} -24x + 24xy - 6xy^2\, dx\, dy$$

$$= \int_{-1}^{0} \left(-12x^2 + 12x^2y - 3x^2y^2\right]_{x=-1-y}^{x=0} dy$$

$$= \int_{-1}^{0} 12(1+y)^2 - 12(1+y)^2 y + 3(1+y)^2 y^2\, dy$$

$$= \int_{-1}^{0} 12 + 12y - 9y^2 - 6y^3 + 3y^4\, dy$$

$$= 12y + 6y^2 - 3y^3 - \frac{3}{2}y^4 + \frac{3}{5}y^5\Big]_{-1}^{0}$$

$$= 0 - \left(-12 + 6 + 3 - \frac{3}{2} - \frac{3}{5}\right) = \frac{51}{10}$$

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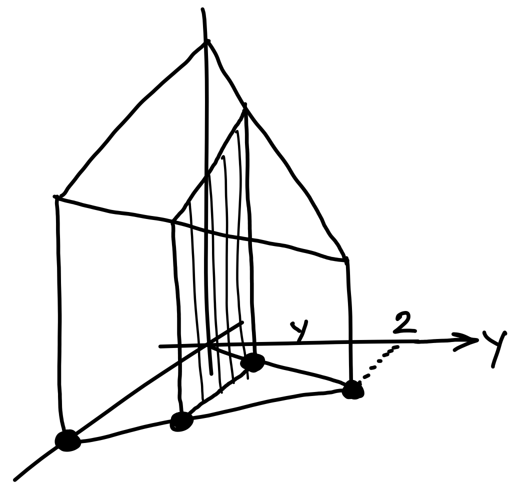
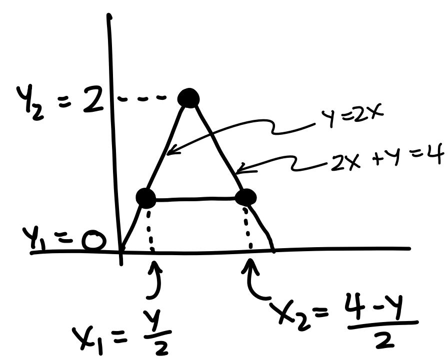
2. *(20 pts)* The region T in xyz-space is bounded by the surfaces $y = 0$, $y = 2x$, $2x + y = 4$, $z = 0$, $x+y+z = 5$, and has a vertex at the origin. Set up $\iiint_T e^x \, dV$ as just one triple iterated integral. (This is possibly with only one of the six choices of orderings of the differentials. You do not have to compute the iterated integral.)

Given the figure, we choose to order the differentials $dz\,dx\,dy$, which requires only 1 iterated integral.



Looking at the projection to the xy-plane, we get the x- and y- bounds as indicated.



With z-bounds coming from the top and bottom surfaces, we have

$$\int_0^2 \int_{\frac{y}{2}}^{\frac{4-y}{2}} \int_0^{5-x-y} e^x \, dz\,dx\,dy$$

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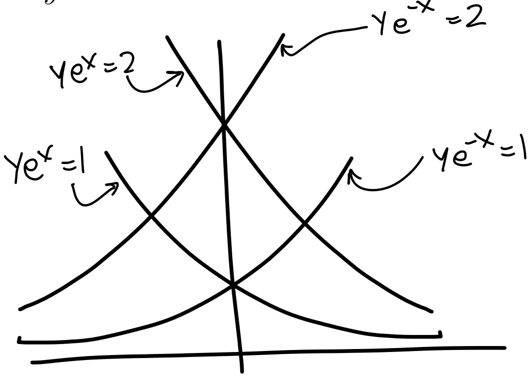
3. *(20 pts)* The region R in the xy-plane is bounded by the curves $y = e^x$, $y = 2e^x$, $y = e^{-x}$, and $y = 2e^{-x}$.

(a) Use a change of variables to rewrite

$$I = \iint_R y^2 e^x - y^2 e^{-x} + 8x^3 y \, dx \, dy$$

as an iterated integral over a rectangle.



Choose $u = ye^{-x}$, $v = ye^x$.

$$\frac{\partial(u,v)}{\partial(x,y)} = \det\begin{pmatrix} -ye^{-x} & e^{-x} \\ ye^x & e^x \end{pmatrix} = -2y$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{-1}{2y} \qquad \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{2y}$$

$$\frac{v}{u} = e^{2x}$$

$$x = \frac{1}{2}\ln\frac{v}{u}$$

$$I = \int_1^2 \int_1^2 \left(y^2e^x - y^2e^{-x} + 8x^3y\right)\left(\frac{1}{2y}\right) du\,dv$$

$$= \int_1^2 \int_1^2 \frac{ye^x - ye^{-x} + 8x^3}{2} \, du\,dv = \int_1^2\int_1^2 \frac{v - u + \left(\ln\left(v/u\right)\right)^3}{2} \, du\,dv$$

(b) Compute the original integral over R by any other means from this course WITHOUT using the above change of variables.

Reflection over y-axis is given by $R(x,y) = (-x,y)$.
Domain R is symmetric, and f is odd because

$$f(R(x,y)) = f(-x,y) = y^2e^{(-x)} - y^2e^{-(-x)} + 8(-x)^3y$$

$$= -\left(y^2e^x - y^2e^{-x} + 8x^3y\right) = -f(x,y)$$

So integral $= 0$ by symmetry.

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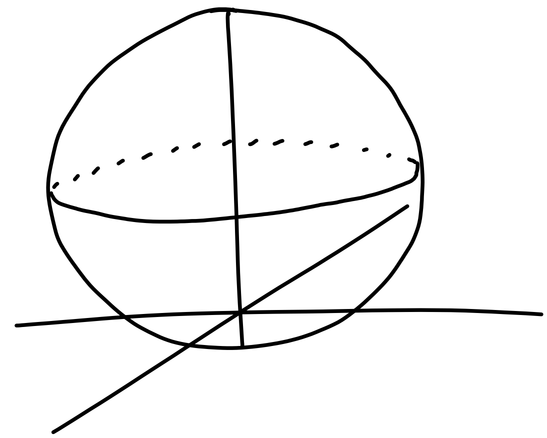
4. *(20 pts)* B is the ball of radius 1 centered at $(0, 0, 1)$. Mass is distributed through B with density at each point in B equal to the distance from that point to the origin. Compute the total mass in B.



Equation of boundary sphere is

$$x^2+y^2+(z-1)^2=1$$

$$x^2+y^2+z^2-2z+1=1$$

$$\rho^2-2\rho\cos\phi=0$$

$$\Rightarrow \rho=2\cos\phi$$

$$m=\iiint \delta\, dV=\iiint \rho\, dV$$

$$=\int_0^{2\pi}\int_0^{\pi/2}\int_0^{2\cos\phi} \rho\ \rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

$$=\int_0^{2\pi}\int_0^{\pi/2}\int_0^{2\cos\phi} \rho^3\sin\phi\, d\rho\, d\phi\, d\theta$$

$$=\int_0^{2\pi}\int_0^{\pi/2}\left(\frac{1}{4}\rho^4\sin\phi\right]_{\rho=0}^{\rho=2\cos\phi} d\phi\, d\theta$$

$$=\int_0^{2\pi}\int_0^{\pi/2} 4\cos^4\phi\sin\phi\, d\phi\, d\theta$$

$$=\int_0^{2\pi}\left(-\frac{4}{5}\cos^5\phi\right]_{\phi=0}^{\phi=\pi/2} d\theta$$

$$=2\pi\cdot\frac{4}{5}=\frac{8\pi}{5}$$

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5. *(20 pts)* A lake bed is the part of the surface $z = x^2 + 4y^2 - 4$ with $z \le 0$ (the water level is at $z = 0$). Algae is growing on the lake bed with density (mass per unit area) depending on the depth below the surface, as given by $\delta = 5e^{3z}$. Set up an iterated integral representing the total mass of algae on the lake bed. (You do not have to compute the iterated integral.)

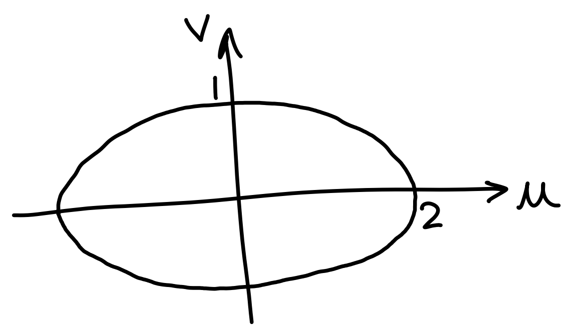
Graph parametrization : $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} u \\ v \\ u^2+4v^2-4 \end{pmatrix}$

$$\vec{X}_u = \begin{pmatrix} 1 \\ 0 \\ 2u \end{pmatrix}, \quad \vec{X}_v = \begin{pmatrix} 0 \\ 1 \\ 8v \end{pmatrix}, \quad \vec{N} = \vec{X}_u \times \vec{X}_v = \begin{pmatrix} -2u \\ -8v \\ 1 \end{pmatrix}$$

$$\|\vec{N}\| = \sqrt{1 + 4u^2 + 64v^2}$$

Pullback domain is bounded by

$z = 0 \Rightarrow u^2 + 4v^2 = 4$:



$$m = \iint_S \delta \, dS = \iint 5e^{3z} \, dS$$

$$= \int_{-1}^{1} \int_{-\sqrt{4-4v^2}}^{\sqrt{4-4v^2}} 5e^{3(u^2+4v^2-4)} \sqrt{1+4u^2+64v^2} \, du \, dv$$

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