EXAM 1

Math 212, 2018 Fall, Clark Bray.

Name: Solutions Section: Student ID:

GENERAL RULES

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

No notes, no books, no calculators. Scratch paper is allowed, but (1) it must be from the instructor, (2) it must be returned with the exam, and (3) it will NOT be graded.

All answers must be reasonably simplified.

All of the policies and guidelines on the class webpages are in effect on this exam.

WRITING RULES

Do not write anything on the QR codes or near the staple.

Use black pen only. You may use a pencil for initial sketches of diagrams, but the final sketch must be drawn over in black pen and you must wipe all erasure residue from the paper.

Work for a given question can ONLY be done on the front or back of the page the question is written on.

DUKE COMMUNITY STANDARD STATEMENT

"I have adhered to the Duke Community Standard in completing this examination."

Signature:

(Nothing on this page will be graded!)

1. *(20 pts)* The line L is parametrized by $\vec{x}(t) = (3 - 2t, 4t + 1, 5t)$, and the line M has symmetric equations $4 - x = y + 3 = z/2$.

   (a) Derive a parametrization of M. (Show the process, don't just write down the result!).

$$4 - x = y + 3 = z/2 = t$$

$$\Rightarrow \begin{aligned} x &= 4 - t \\ y &= -3 + t \\ z &= 2t \end{aligned} \quad \Rightarrow \boxed{\vec{x} = \begin{pmatrix} 4 \\ -3 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}}$$

   (b) Find the equation of the unique plane P that contains L and is parallel to M.

L is parametrized by $\vec{x} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -2 \\ 4 \\ 5 \end{pmatrix}$.

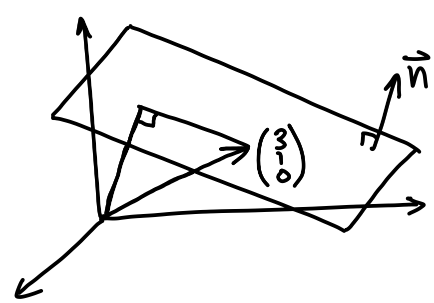
Choose $\vec{n} = \begin{pmatrix} -2 \\ 4 \\ 5 \end{pmatrix} \times \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$ and $\vec{x}_0 = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$.

The equation of the plane is $\vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{x}_0$

$$\boxed{3x - y + 2z = 8}$$

   (c) Find the distance from P to the origin by way of an argument relating this distance to a component of $\vec{x}$ (in P) in some direction. (Do not just cite a formula from the homework!)

$$\begin{aligned} \text{dist.} &= \text{comp}_{\vec{n}} \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} \cdot \vec{n} \Big/ \|\vec{n}\| \\ &= 8 \Big/ \sqrt{3^2 + (-1)^2 + 2^2} \\ &= 8 / \sqrt{14} \end{aligned}$$



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2. *(20 pts)* A particle is moving with velocity given by $\vec{v}(t) = (4\sin t - 3\cos t, e^{2t})$. At time $t = 0$ this particle was at $\vec{x}_0 = (5, 6)$.

(a) Find an expression for the position $\vec{x}(t)$ of this particle.

$$\vec{x}(t) = \int \vec{v}(t)\,dt = \int \begin{pmatrix} 4\sin t - 3\cos t \\ e^{2t} \end{pmatrix} dt$$

$$= \begin{pmatrix} -4\cos t - 3\sin t \\ \frac{1}{2}e^{2t} \end{pmatrix} + \vec{c}$$

At $t=0$,

$$\begin{pmatrix} 5 \\ 6 \end{pmatrix} = \begin{pmatrix} -4 \\ \frac{1}{2} \end{pmatrix} + \vec{c} \quad \Rightarrow \vec{c} = \begin{pmatrix} 9 \\ 11/2 \end{pmatrix}$$

$$\Rightarrow \vec{x}(t) = \begin{pmatrix} -4\cos t - 3\sin t \\ \frac{1}{2}e^{2t} \end{pmatrix} + \begin{pmatrix} 9 \\ 11/2 \end{pmatrix}$$

(b) Find the curvature of this particle's path at the moment $t = 0$.

$$\vec{v}(0) = \begin{pmatrix} -3 \\ 1 \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$\vec{a}(t) = \vec{v}\,' = \begin{pmatrix} 4\cos t + 3\sin t \\ 2e^{2t} \end{pmatrix} \quad \text{so} \quad \vec{a}(0) = \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} x'' \\ y'' \end{pmatrix}$$

Then

$$\kappa(0) = \frac{|x'y'' - x''y'|}{v^3} = \frac{|(-3)(2) - (4)(1)|}{\left((-3)^2 + (1)^2\right)^{3/2}} = \frac{10}{10^{3/2}} = \frac{1}{\sqrt{10}}$$

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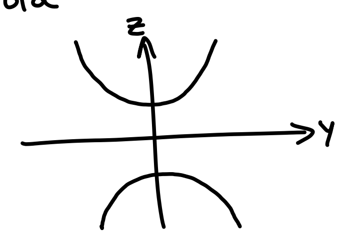
3. *(20 pts)* The surface S has equation $z^2 - x^2 - y^2 = 1$.

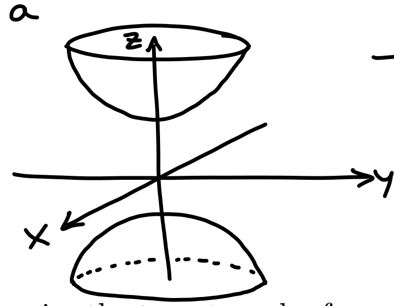
(a) What does S look like? Explain in as much detail as you can; feel free to include a drawing if that might help your explanation.

$z^2-(x^2+y^2)=1$ has rotational symmetry around the z-axis.

Cross section in $x=0$ plane is $z^2-y^2=1$, a hyperbola:



Rotating around the z-axis gives a hyperboloid of two sheets:



(b) Is S the graph of a function? If so, give the domain, the target, and a formula for evaluating this function. If not, explain why not.

S is <u>not</u> a graph, as it fails the vertical line test in all directions.

(c) Is S a level set of a function? If so, give the domain, the target, and a formula for evaluating this function. If not, explain why not.

$z^2-x^2-y^2=1$ is the $g=1$ level set of the function $g:\mathbb{R}^3 \to \mathbb{R}^1$ evaluated by

$$g(x,y,z) = z^2-x^2-y^2$$

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4. *(20 pts)*

(a) Find the equation of the tangent plane to the graph of the function $g(x, y) = x^2 - y + 3$ at the point $(1, 1, 3)$.

$$\frac{\partial g}{\partial x} = 2x \qquad \frac{\partial g}{\partial y} = -1.$$

$$\left.\frac{\partial g}{\partial x}\right|_{(1,1)} = 2 \qquad \left.\frac{\partial g}{\partial y}\right|_{(1,1)} = -1.$$

Equation of tangent plane is the graph $z = L(x,y)$ of the linear approximation,

$$\boxed{z = 3 + 2(x-1) - 1(y-1)}$$

(b) The surface S is the graph $z = f(x, y)$ of the function $f : \mathbb{R}^2 \to \mathbb{R}$. At the point $(1, 2, 3)$ on S the equation of the tangent plane to S is $4x - 5y + 3z = 3$. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

Tangent plane $z = \frac{3-4x+5y}{3}$ is graph $z = L(x,y)$ of

$$L(x,y) = \frac{3-4x+5y}{3}$$

This has the same partials as f, so

$$\frac{\partial f}{\partial x} = -\frac{4}{3} \qquad \frac{\partial f}{\partial y} = \frac{5}{3}$$

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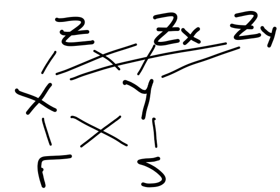
5. *(20 pts)* The variable z is computed as a twice continuously differentiable function of x and y, and x and y are computed from r and s by $x = 2r + 3s$ and $y = 5r + 7s$. Compute $\frac{\partial^2 z}{\partial r \partial s}$ in terms of the partials of z.

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$= 3z_x + 7z_y$$



$$\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial s}\right) = \frac{\partial}{\partial r}\left(3z_x + 7z_y\right)$$

$$= 3\left(\frac{\partial z_x}{\partial r}\right) + 7\left(\frac{\partial z_y}{\partial r}\right)$$

$$= 3\left(\frac{\partial z_x}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z_x}{\partial y}\frac{\partial y}{\partial r}\right) + 7\left(\frac{\partial z_y}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z_y}{\partial y}\frac{\partial y}{\partial r}\right)$$

$$= 3\left(2z_{xx} + 5z_{xy}\right) + 7\left(2z_{xy} + 5z_{yy}\right)$$

$$= 6z_{xx} + 29z_{xy} + 35z_{yy}$$

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