EXAM 3

Math 212, 2017-2018 Spring, Clark Bray.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

Solutions

No notes, no books, no calculators.

All answers must be simplified.

All of the policies and guidelines on the class webpages are in effect on this exam.

Do not write anything on the QR codes or near the staple.

Work for a given question can ONLY be done on the front or back of the page the question is written on.

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ____________________

(Nothing on this page will be graded!)

1. *(20 pts)* Position on the side of a particular hill is given by distance (in meters) east (x) and north (y) of a fixed reference point. A combination of wind and steepness exerts a force (in Newtons) on you at a point (x, y) given by $\vec{F}(x, y) = (4 + y + 3x^2, 1 + x + 6y^2)$.

The path you walk, from $t = 0$ to $t = 1$, is parametrized by $\vec{x}(t) = (10\cos(\pi t), 5\sin(3\pi t/2))$. Compute the amount of work (in Joules, aka Newton-meters) that you perform in this walk.

$\text{grn}\,\vec{F} = Q_x - P_y = 1 - 1 = 0$. So $\vec{F} = \nabla f$ for some f.

$$f = \int 4 + y + 3x^2\,dx = 4x + xy + x^3 + c_1(y)$$
$$f = \int 1 + x + 6y^2\,dy = y + xy + 2y^3 + c_2(x)$$

These equations are resolved with $f = xy + 4x + x^3 + y + 2y^3$

Then

$$W = -\int \vec{F} \cdot d\vec{x} = -\left(f(\vec{x}(1)) - f(\vec{x}(0))\right)$$
$$= -\left(f(-10, -5) - f(10, 0)\right)$$
$$= -\left((50\ 40 - 1000 - 5 - 250) - (0 + 40 + 1000 + 0 + 0)\right)$$
$$= 2285$$

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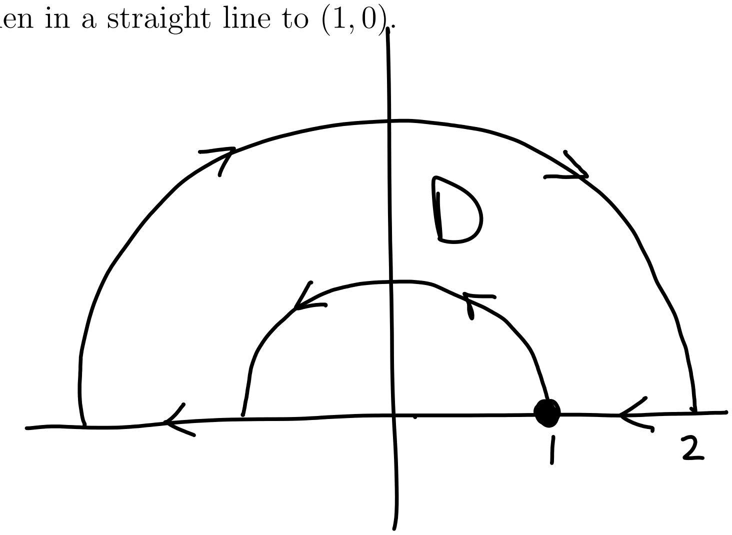
2. *(20 pts)* The circle C_1 is centered at the origin with radius 1, and the circle C_2 is centered at the origin with radius 2.

Compute the line integral $\int_P \vec{G} \cdot d\vec{x}$ of the vector field $\vec{G}(x, y) = (3x^2 + 2y, 5x - 6y^2)$ over the path P that starts at $(1, 0)$, moves counter clockwise around C_1 to $(-1, 0)$, then in a straight line to $(-2, 0)$, then clockwise around C_2 to $(2, 0)$, and then in a straight line to $(1, 0)$.

P is closed and bounds the region D, as drawn here.



$$P = -\partial D$$

Then $\int_P \vec{G} \cdot d\vec{x} = -\int_{\partial D} \vec{G} \cdot d\vec{x}$

$$= -\iint_D \operatorname{grn} \vec{G} \, dA$$

$$= -\iint 3 \, dA$$

$$= (-3)(\text{area of } D)$$

$$= (-3)\left(\tfrac{1}{2}\left(\pi \cdot 2^2 - \pi \cdot 1^2\right)\right)$$

$$= -\tfrac{9}{2}\pi$$

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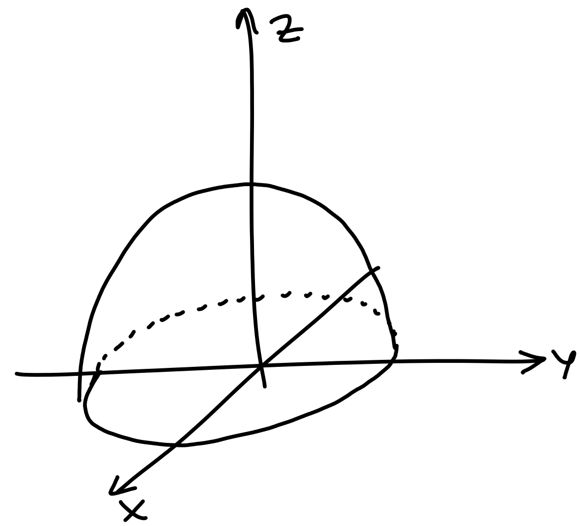
3. *(20 pts)* Compute the flux of the vector field $\vec{F}(\vec{x}) = (3x+4y, 5x-z, x^3+yz)$ through the surface S that is the inward oriented boundary of the solid R defined by $\rho \leq 2$ and $z \geq 0$.

$S = -\partial R$



$$\Phi = \iint_S \vec{F} \cdot d\vec{S}$$

$$= -\iint_{\partial R} \vec{F} \cdot d\vec{S}$$

$$= -\iiint_R \nabla \cdot \vec{F} \, dV$$

$$= -\iiint_R (3+y) \, dV$$

$$= \iiint_R (-3) \, dV - \iiint_R y \, dV$$

R is symmetric through the xz-plane with reflection

$$L(x,y,z) = (x,-y,z)$$

And $f(x,y,z) = y$ has odd symm. because

$$f(L(x,y,z)) = f(x,-y,z)$$
$$= -y$$
$$= -f(x,y,z)$$

So $\int = 0$ by symmetry.

$$= (-3)(\text{volume of } R) - 0$$

$$= (-3)\left(\frac{1}{2} \cdot \frac{4}{3}\pi \cdot 2^3\right)$$

$$= -16\pi$$

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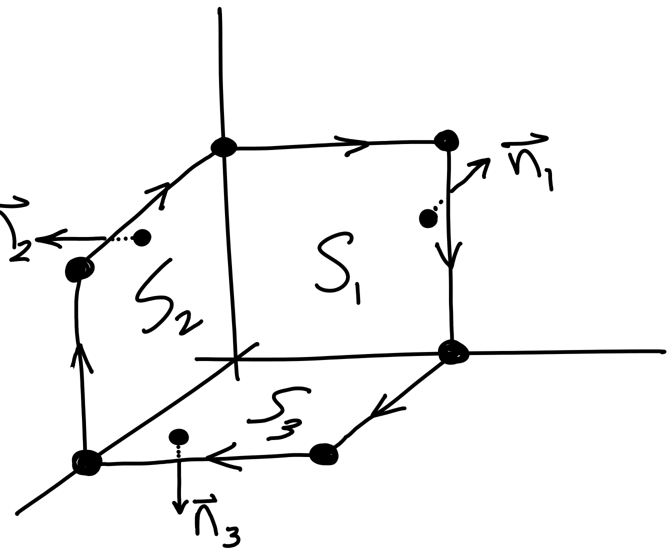
4. *(20 pts)* The curve C starts at $(0,0,1)$, and then in straight line segments goes to $(0,1,1)$, $(0,1,0)$, $(1,1,0)$, $(1,0,0)$, $(1,0,1)$, and then back to $(0,0,1)$. Use Stokes's curl theorem to compute the circulation along C of the vector field $\vec{F}(\vec{x}) = (x^3 + xyz, y^3 + xyz, z^3 + xyz)$.

$C = \partial S$, where S consists of the three surfaces S_1, S_2, S_3 as drawn here.



Then

$$\int_C \vec{F} \cdot d\vec{x} = \iint_S (\nabla \times \vec{F}) \cdot d\vec{S}$$

$$= \iint_S \begin{pmatrix} xz - xy \\ xy - yz \\ yz - xz \end{pmatrix} \cdot d\vec{S}$$

$$= \iint_{S_1} \begin{pmatrix} xz - xy \\ xy - yz \\ yz - xz \end{pmatrix} \cdot \vec{n}_1 \, dS \quad \vec{n}_1 = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$$

$$+ \iint_{S_2} \begin{pmatrix} xz - xy \\ xy - yz \\ yz - xz \end{pmatrix} \cdot \vec{n}_2 \, dS \quad \vec{n}_2 = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$$

$$+ \iint_{S_3} \begin{pmatrix} xz - xy \\ xy - yz \\ yz - xz \end{pmatrix} \cdot \vec{n}_3 \, dS \quad \vec{n}_3 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

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$$= \iint_{S_1} xy - xz \, dS \quad \text{(where } x=0\text{)}$$

$$+ \iint_{S_2} yz - xy \, dS \quad \text{(where } y=0\text{)}$$

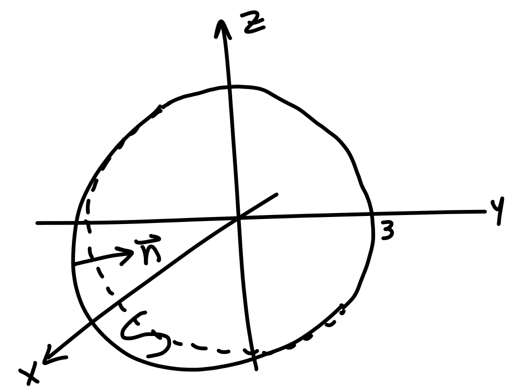
$$+ \iint_{S_3} xz - yz \, dS \quad \text{(where } z=0\text{)}$$

$$= 0$$

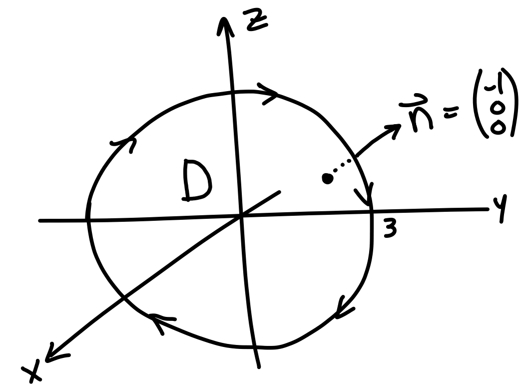
5. *(20 pts)* The surface S is the part of the sphere $\rho = 3$ with $x \geq 0$, oriented toward the origin. Compute the flux through S of the vector field $\vec{H}(\vec{x}) = (2 + x^4 - 5y, x^2 + 3z, -4x^3 z)$.

$$\nabla \cdot \vec{H} = (4x^3) + (0) + (-4x^3) = 0$$

So $\vec{H}$ is surface independent.



The disk D in the yz-plane has the same boundary as S, so



$$\iint_S \vec{H} \cdot d\vec{S} = \iint_D \vec{H} \cdot d\vec{S}$$

$$= \iint_D \vec{H} \cdot \vec{n} \, dS$$

$$= \iint_D -2 - x^4 + 5y \, dS$$

$$= \iint_D (-2)\, dS + \iint_D (-x^4)\, dS + \iint_D 5y \, dS$$

$$= -2(\text{area of } D) + 0 + 0$$

(since $x=0$ on D)

(D is symmetric through the xz-plane with $R(x,y,z) = (x,-y,z)$, and $f = 5y$ is odd since $f(R(x,y,z)) = f(x,-y,z) = -5y = -f(x,y,z)$)

$$= (-2)(\pi \cdot 3^2)$$

$$= -18\pi$$

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