EXAM 2

Math 212, 2017-2018 Spring, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.
All answers must be simplified. All of the policies and guidelines
on the class webpages are in effect on this exam.

Good luck!

Name Solutions

"I have adhered to the Duke Community
Standard in completing this
examination."

Signature: ______________________

1. __________

2. __________

3. __________

4. __________

5. __________

Total Score __________ (/100 points)

1. *(20 pts)* Let f be a function of the rectangular coordinates x, y, and z, and let ρ, ϕ, and θ be the usual spherical coordinates.

   (a) Find an expression for $\dfrac{\partial f}{\partial \rho}$ in terms of the rectangular partials of f and the spherical coordinates.

$$\begin{tikzcd} \rho \arrow[r] \arrow[dr] \arrow[ddr] & x \arrow[dr] & \\ \phi & y \arrow[r] & f \\ \theta & z \arrow[ur] & \end{tikzcd}$$

$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta$$
$$z = \rho \cos\phi$$

$$\frac{\partial f}{\partial \rho} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \rho} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \rho} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial \rho}$$

$$= f_x \sin\phi\cos\theta + f_y \sin\phi\sin\theta + f_z\cos\phi$$

   (b) Find an expression for $\dfrac{\partial^2 f}{\partial \rho^2}$ in terms of the rectangular partials of f and the spherical coordinates.

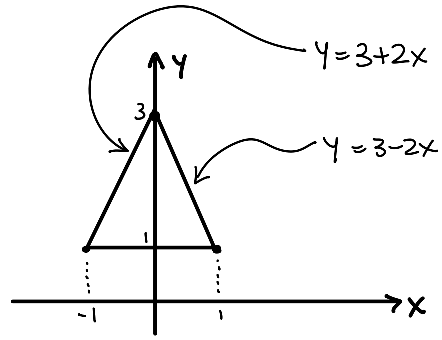
$$\frac{\partial^2 f}{\partial \rho^2} = \frac{\partial}{\partial \rho}\left(f_x \sin\phi\cos\theta + f_y\sin\phi\sin\theta + f_z\cos\phi\right)$$

$$= \left(\frac{\partial f_x}{\partial \rho}\right)(\sin\phi\cos\theta) + \left(\frac{\partial f_y}{\partial \rho}\right)(\sin\phi\sin\theta) + \left(\frac{\partial f_z}{\partial \rho}\right)\cos\phi$$

$$= \left(\frac{\partial f_x}{\partial x}\sin\phi\cos\theta + \frac{\partial f_x}{\partial y}\sin\phi\sin\theta + \frac{\partial f_x}{\partial z}\cos\phi\right)(\sin\phi\cos\theta)$$
$$+\left(\frac{\partial f_y}{\partial x}\sin\phi\cos\theta + \frac{\partial f_y}{\partial y}\sin\phi\sin\theta + \frac{\partial f_y}{\partial z}\cos\phi\right)(\sin\phi\sin\theta)$$
$$+\left(\frac{\partial f_z}{\partial x}\sin\phi\cos\theta + \frac{\partial f_z}{\partial y}\sin\phi\sin\theta + \frac{\partial f_z}{\partial z}\cos\phi\right)(\cos\phi)$$

$$= f_{xx}\sin^2\phi\cos^2\theta + f_{yy}\sin^2\phi\sin^2\theta + f_{zz}\cos^2\phi$$
$$2f_{xy}\sin^2\phi\sin\theta\cos\theta + 2f_{yz}\sin\phi\cos\phi\sin\theta + 2f_{xz}\sin\phi\cos\phi\cos\theta$$

2. *(20 pts)* The solid triangle T in the xy-plane has vertices $(-1,1)$, $(0,3)$, and $(1,1)$. A total mass of $m = 10/3$ is distributed over this area as indicated by the density function $\delta = y$. Compute the coordinates $\bar{x}$ and $\bar{y}$ of the centroid of this mass.



$$\bar{x} = \frac{1}{m}\iint_T x\,\delta\,dA = \frac{3}{10}\iint_T xy\,dA$$

Domain is symmetric over the y-axis.

$R(x,y) = (-x,y)$

$f(-x,y) = -xy = -f(x,y)$ so f is odd.

So $\bar{x} = 0$.

$$\bar{y} = \frac{1}{m}\iint_T y\,\delta\,dA = \frac{3}{10}\int_1^3 \int_{\frac{y-3}{2}}^{\frac{3-y}{2}} y^2\,dx\,dy$$

$$= \frac{3}{10}\int_1^3 xy^2\Big]_{x=\frac{y-3}{2}}^{x=\frac{3-y}{2}} dy$$

$$= \frac{3}{10}\int_1^3 3y^2 - y^3\,dy$$

$$= \frac{3}{10}\left(y^3 - \frac{1}{4}y^4\right]_1^3$$

$$= \frac{3}{10}\left(\left(27 - \frac{81}{4}\right) - \left(1 - \frac{1}{4}\right)\right)$$

$$= \frac{3}{10}(6)$$

$$= \frac{9}{5}$$

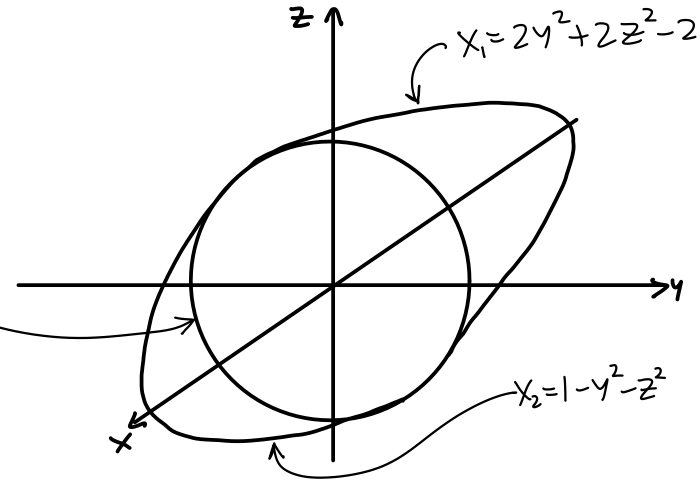
3. *(20 pts)* The solid R is bounded by the surfaces with equations $x - 2y^2 - 2z^2 + 2 = 0$ and $x + y^2 + z^2 = 1$. Mass is distributed in this solid with density given by $\delta(x, y, z) = e^{x-y}$.

Set up as an iterated triple integral the moment of inertia of this solid around the y-axis. (You do NOT have to compute the integral.)

Intersection is where $x_1 = x_2$,

$$2y^2 + 2z^2 - 2 = 1 - y^2 - z^2$$

$$\Rightarrow y^2 + z^2 = 1$$



$$I = \iiint_R r^2 \, dm = \iiint_R r^2 \delta \, dV$$

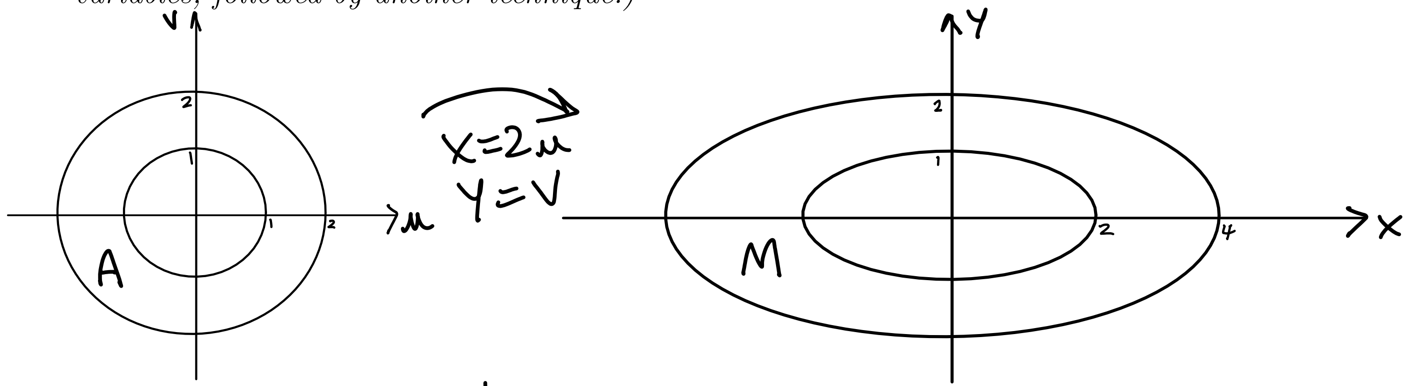
$$= \iiint_R (x^2 + z^2) e^{x-y} \, dV$$

$$= \int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_{2y^2+2z^2-2}^{1-y^2-z^2} (x^2 + z^2) e^{x-y} \, dx \, dz \, dy$$

4. *(20 pts)* The ellipses E_1 and E_2 in the xy-plane have equations below.

$$E_1 : \left(\frac{x}{2}\right)^2 + \left(\frac{y}{1}\right)^2 = 1 \quad \text{and} \quad E_2 : \left(\frac{x}{4}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$$

The region between these two ellipses is M. Compute $\iint_M x^2\,dx\,dy$. *(Hint: Use a change of variables, followed by another technique.)*



$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\det\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\right| = 2, \quad \text{so } dx\,dy = 2\,du\,dv$$

Then

$$\iint_M x^2\,dx\,dy = \iint_A (2u)^2\, 2\,du\,dv = \iint_A 8u^2\,du\,dv$$

Using polar coordinates for this integral on A, we get

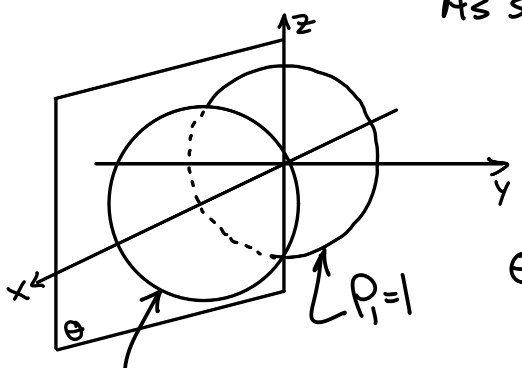
$$\int_0^{2\pi}\int_1^2 8(r\cos\theta)^2\, r\,dr\,d\theta = \int_0^{2\pi}\int_1^2 8r^3\cos^2\theta\,dr\,d\theta$$

$$= \int_0^{2\pi} 2r^4\cos^2\theta\Big]_{r=1}^{r=2}\,d\theta = 30\int_0^{2\pi}\cos^2\theta\,d\theta = 15\int_0^{2\pi} 1 + \cos 2\theta\,d\theta$$

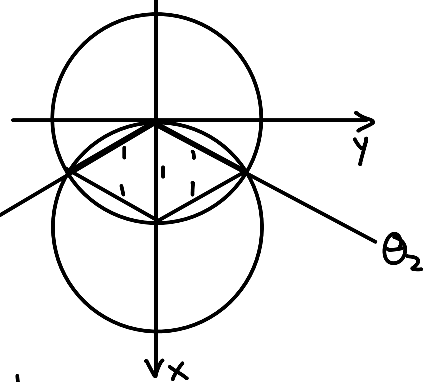
$$= 15\cdot 2\pi = 30\pi$$

5. *(20 pts)* The sphere S_1 is the unit sphere centered at the origin, and the sphere S_2 is the unit sphere centered at $(1, 0, 0)$. The solid D consists of the points that are outside of S_1 and inside of S_2.

Set up $\iiint_D x\, dV$ as an iterated triple integral in spherical coordinates. (You do NOT have to compute the integral.)



As seen from above:

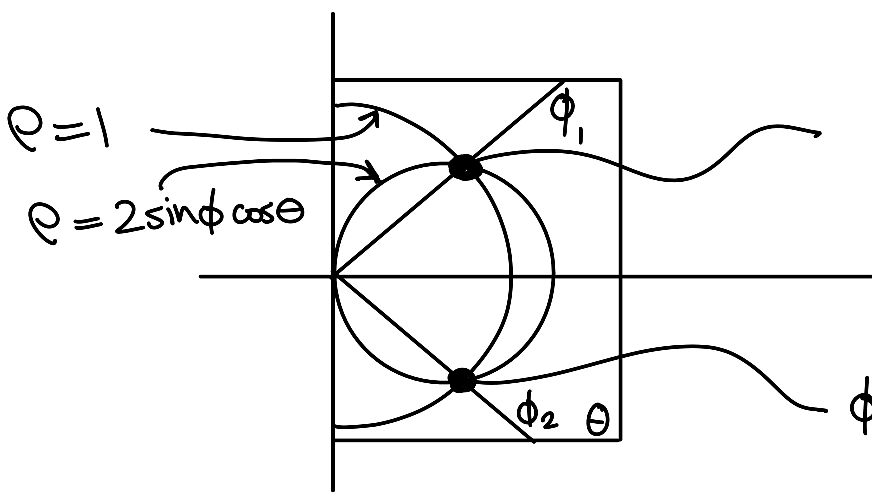


θ ranges from $-\pi/3$ to $\pi/3$

$$(x-1)^2 + y^2 + z^2 = 1$$

$$x^2 - 2x + 1 + y^2 + z^2 = 1$$

$$\rho^2 - 2\rho \sin\phi \cos\theta = 0 \implies \rho_2 = 2\sin\phi\cos\theta$$



$$1 = 2\sin\phi\cos\theta$$

$$\phi_1 = \arcsin\left(\frac{1}{2\cos\theta}\right)$$

$$\phi_2 = \pi - \phi_1 = \pi - \arcsin\left(\frac{1}{2\cos\theta}\right)$$

So

$$\iiint_D x\, dV$$

$$= \int_{-\pi/3}^{\pi/3} \int_{\arcsin\left(\frac{1}{2\cos\theta}\right)}^{\pi - \arcsin\left(\frac{1}{2\cos\theta}\right)} \int_1^{2\sin\phi\cos\theta} (\rho\sin\phi\cos\theta)\, \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$