EXAM 1

Math 212, 2017-2018 Spring, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.
All answers must be simplified. All of the policies and guidelines
on the class webpages are in effect on this exam.

Good luck!

Name Solutions

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ______________________

1. __________

2. __________

3. __________

4. __________

5. __________

Total Score __________ (/100 points)

1. *(20 pts)*

(a) Compute the cross product of $\vec{v} = (1, 2, 3)$ and $\vec{w} = (2, 0, 1)$.

$$\vec{v} \times \vec{w} = \det\begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ 1 & 2 & 3 \\ 2 & 0 & 1 \end{pmatrix}$$

$$= (2, 5, -4)$$

(b) Find the area of the parallelogram defined by $\vec{v}$ and $\vec{w}$.

$$\text{area} = \|\vec{v} \times \vec{w}\| = \sqrt{2^2 + 5^2 + (-4)^2}$$

$$= \sqrt{45} = 3\sqrt{5}$$

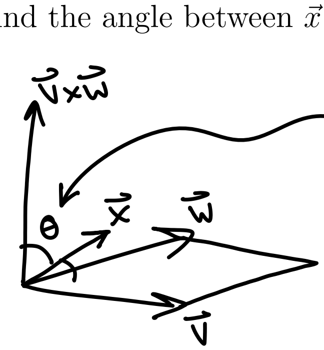
(c) Make explicit use of the cross product computed in part (a) to find the volume of the parallelepiped defined by $\vec{v}$, $\vec{w}$, and $\vec{x} = (3, 3, 5)$ and to decide if the listing $\vec{v}$, $\vec{w}$, $\vec{x}$ is in right or left hand order.

$$\det\begin{pmatrix} \text{---} \; \vec{x} \; \text{---} \\ \text{---} \; \vec{v} \; \text{---} \\ \text{---} \; \vec{w} \; \text{---} \end{pmatrix} = \vec{x} \cdot (\vec{v} \times \vec{w}) = (3,3,5) \cdot (2,5,-4) = 1$$

$$\text{volume} = |\det| = 1$$

order of $\vec{v}, \vec{w}, \vec{x}$ = order of $\vec{x}, \vec{v}, \vec{w}$ is RHO b/c det > 0.

(d) Find the angle between $\vec{x}$ and the parallelogram defined by $\vec{v}$ and $\vec{w}$.



$$\theta = \arccos\left(\frac{\vec{x} \cdot (\vec{v} \times \vec{w})}{\|\vec{x}\| \, \|\vec{v} \times \vec{w}\|}\right) = \arccos\left(\frac{1}{\sqrt{43} \cdot 3\sqrt{5}}\right)$$

$$= \arccos\left(\frac{1}{3\sqrt{215}}\right)$$

$$\text{angle} = \frac{\pi}{2} - \theta = \frac{\pi}{2} - \arccos\left(\frac{1}{3\sqrt{215}}\right)$$

2. *(20 pts)* Find the point of intersection of the plane with equation $2x - 4y + 3z = 1$ and the line parametrized by $(x, y, z) = (2 + t, 1 - 5t, 3 + 2t)$.

$$2(2+t) - 4(1-5t) + 3(3+2t) = 1$$

$$9 + 28t = 1$$

$$t = \frac{-8}{28} = \frac{-2}{7}$$

Point of intersection is

$$\vec{x}\left(-2/7\right) = \begin{pmatrix} 2+\left(\frac{-2}{7}\right) \\ 1-5\left(\frac{-2}{7}\right) \\ 3+2\left(\frac{-2}{7}\right) \end{pmatrix} = \begin{pmatrix} 12/7 \\ 17/7 \\ 17/7 \end{pmatrix}$$

3. *(20 pts)* A dog runs through a field from $t = 0$ to $t = 2\pi$ with his position parametrized by

$$\vec{x}(t) = \begin{pmatrix} t + \cos t \\ \sin t \end{pmatrix}$$

(a) Compute expressions for the dog's velocity and the acceleration.

$$\vec{v} = \vec{x}' = \begin{pmatrix} 1 - \sin t \\ \cos t \end{pmatrix}$$

$$\vec{a} = \vec{v}' = \begin{pmatrix} -\cos t \\ -\sin t \end{pmatrix}$$

(b) At what time is the dog moving the fastest?

$\|\vec{v}\|^2 = (1-\sin t)^2 + (\cos t)^2 = 1 - 2\sin t + \sin^2 t + \cos^2 t = 2 - 2\sin t$

Dog is moving fastest at max. value of $\|\vec{v}\|^2$, which is attained when $\sin t = -1$, when $t = \frac{3\pi}{2}$.

(c) Set up, but do not evaluate, an integral to compute the total distance the dog runs.

$v = \|\vec{v}\| = \sqrt{2 - 2\sin t}$

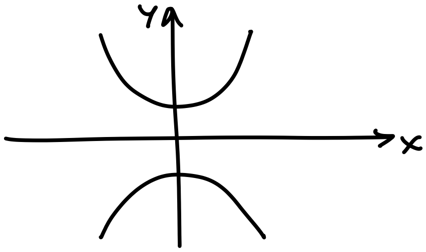
distance = arc length

$= \int_0^{2\pi} v \, dt$

$= \int_0^{2\pi} \sqrt{2 - 2\sin t} \, dt$

4. *(20 pts)* Bob makes a surface S by starting with the curve C in the xy-plane with equation $x^2 - y^2 + 1 = 0$, stretching by a factor of 3 in the x-direction, shifting in the negative y-direction by a distance of 2, and then rotating the result around the y-axis.

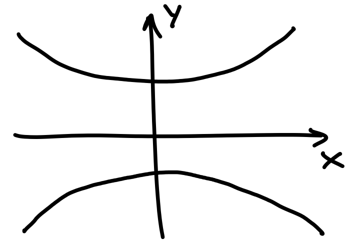
What is the equation for the surface S? Explain with figures how you know if it is a sphere, ellipsoid, paraboloid, hyperboloid of one sheet, hyperboloid of two sheets, or hyperbolic paraboloid.



$x^2 - y^2 + 1 = 0$

stretch by $\cdot 3$ in x-dir.

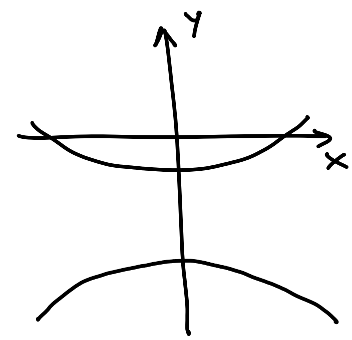
"x" $\mapsto$ "$\frac{x}{3}$"



$\left(\frac{x}{3}\right)^2 - y^2 + 1 = 0$

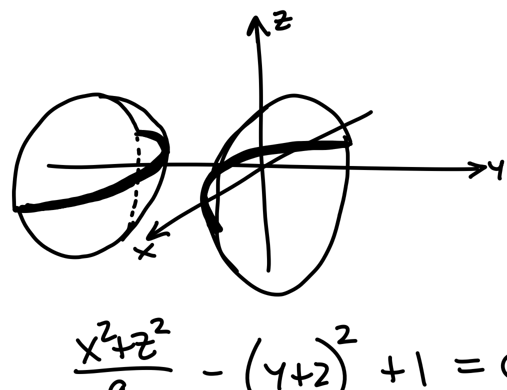
shift by -2 in y-dir.

"y" $\mapsto$ "$y+2$"



$\frac{x^2}{9} - (y+2)^2 + 1 = 0$

The result of the rotation would be rotationally symmetric around the y-axis (so x, z appear only as part of x^2+z^2), and the cross section in $z=0$ is the above equation. So



$$\frac{x^2+z^2}{9} - (y+2)^2 + 1 = 0$$

As shown in these figures this is a hyperboloid of 2 sheets.

5. *(20 pts)* The plane P has equation $x - 2y - 3z = 7$.

(a) Find a function g whose graph is P, and note specifically the domain and the target of g.

$$z = \frac{x-2y-7}{3}$$

This is the graph of $g : \mathbb{R}^2 \to \mathbb{R}^1$ defined by $g(x,y) = \frac{x-2y-7}{3}$

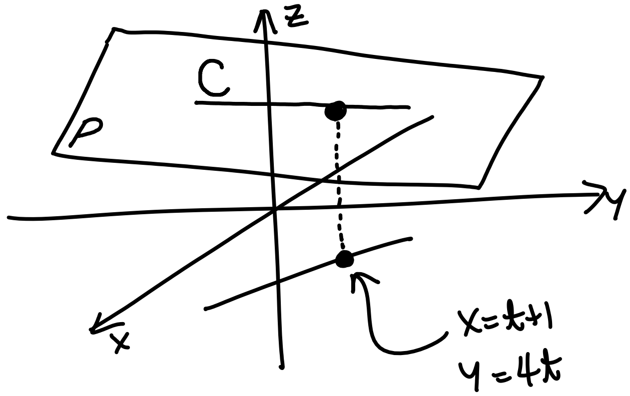
(b) Find a function L one of whose level sets is P, and note specifically the domain and the target of L.

$x - 2y - 3z = 7$

This is the $L=7$ level set of the function $L : \mathbb{R}^3 \to \mathbb{R}^1$

defined by $L(x,y,z) = x - 2y - 3z$

(c) The line C in P has its shadow in the xy-plane parametrized by $(t+1, 4t)$. Find a parametrization $\vec{x}$ of the line C, and note specifically the domain and the target of $\vec{x}$.



$x(t)$ and $y(t)$ are given by the shadow.

In the plane, $z = g(x,y)$, so

$$z(t) = \frac{(t+1) - 2(4t) - 7}{3} = \frac{-7t-6}{3}$$

So $\vec{x} : \mathbb{R}^1 \to \mathbb{R}^3$ is given by $\vec{x}(t) = \left(t+1, 4t, \frac{-7t-6}{3}\right)$.