

# EXAM 3

Math 212, 2017-2018 Fall, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING  
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

All answers must be simplified. All of the policies and guidelines  
on the class webpages are in effect on this exam.

Good luck!

Name Solutions

“I have adhered to the Duke Community  
Standard in completing this  
examination.”

1. \_\_\_\_\_

2. \_\_\_\_\_

3. \_\_\_\_\_

4. \_\_\_\_\_

5. \_\_\_\_\_

Signature: \_\_\_\_\_

Total Score \_\_\_\_\_ (/100 points)

1. (20 pts) The line  $L$  is the intersection of the planes  $x = 2y$  and  $x = 3z$ , and the downward oriented curve  $C$  is the part of this line that is in the ball centered at the origin and of radius 7. Find the line integral over this curve  $C$  of the vector field  $\vec{F}$  below.

$$\vec{F}(x, y, z) = (yz, xz + 3z, xy + 3y)$$

$$\nabla \times \vec{F} = (0, 0, 0) \Rightarrow \vec{F} \text{ is a gradient}$$

$$\left. \begin{aligned} f &= \int P dx = xyz + c_1(y, z) \\ f &= \int Q dy = xyz + 3yz + c_2(x, z) \\ f &= \int R dz = xyz + 3yz + c_3(x, y) \end{aligned} \right\} \Rightarrow f = xyz + 3yz \text{ is an antigradient.}$$

Endpoints of  $C$  are intersections of  $x=2y$ ,  $x=3z$ ,

and  $x^2 + y^2 + z^2 = 7^2$

$$\Rightarrow x^2 + \left(\frac{x}{2}\right)^2 + \left(\frac{x}{3}\right)^2 = 7^2$$

$$x^2 \left(1 + \frac{1}{4} + \frac{1}{9}\right) = 49$$

$$x^2 \left(\frac{36 + 9 + 4}{36}\right) = 49$$

$$x = \pm 6$$

endpoints are

$$\begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} -6 \\ -3 \\ -2 \end{pmatrix}$$

start  $\nearrow$

end  $\nearrow$

(because of downward orientation.)

Then

$$\begin{aligned} \int_C \vec{F} \cdot d\vec{x} &= f(\vec{b}) - f(\vec{a}) \\ &= f\left(\begin{pmatrix} -6 \\ -3 \\ -2 \end{pmatrix}\right) - f\left(\begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix}\right) \end{aligned}$$

$$= (-36 + 18) - (36 + 18) = \boxed{-72}$$

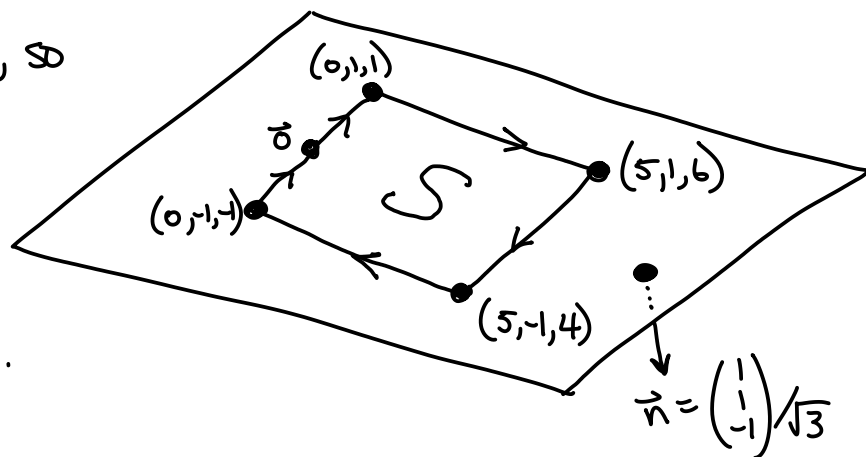
2. (20 pts) Compute the line integral of the vector field  $\vec{G}$  (below) around the polygonal curve  $P$  that starts at the origin, then goes in sequence to  $(0, 1, 1)$ ,  $(5, 1, 6)$ ,  $(5, -1, 4)$ ,  $(0, -1, -1)$ , and then back to the origin.

$$\vec{G}(x, y, z) = \begin{pmatrix} x + 2z \\ 3x - y \\ 2y - 5z \end{pmatrix}$$

(Hint: All of the edges of  $P$  are orthogonal to the vector  $(1, 1, -1)$ .)

$\Rightarrow P$  is contained in a plane with equation  $|x+y-z=c$ .  
Checking points shows it is  $x+y-z=0$ .

This is a closed curve, so it is the boundary of the surface  $S$  it encloses in the plane  $x+y-z=0$ .



Stokes's theorem then gives us

$$\int_P \vec{G} \cdot d\vec{x} = \int_{\partial S} \vec{G} \cdot d\vec{x} = \iint_S (\nabla \times \vec{G}) \cdot d\vec{S}$$

$$= \iint_S \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} / \sqrt{3} \, dS$$

$$= \left( \frac{1}{\sqrt{3}} \right) (\text{area of } S)$$

$$= \left( \frac{1}{\sqrt{3}} \right) \left\| \begin{pmatrix} 5 \\ 0 \\ 5 \end{pmatrix} \times \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} \right\|$$

$\left( S \text{ is a parallelogram} \right)$

$$= \left( \frac{1}{\sqrt{3}} \right) \left\| \begin{pmatrix} -10 \\ -10 \\ 10 \end{pmatrix} \right\| = \boxed{10}$$

3. (20 pts) The curve  $G$  is parametrized by  $\vec{x}(t) = (\sin t, \cos t, t)$  with  $t \in [0, \pi]$ . Find the line integral over  $G$  of the vector field  $\vec{H}$  below.

$$\vec{H}(x, y, z) = (x, x, x^2 y)$$

$$\nabla \times \vec{H} = (x^2, \cdot, \cdot) \neq 0 \Rightarrow \vec{H} \text{ is not a gradient.}$$

$$\vec{x}(0) = (0, 1, 0) \neq \vec{x}(\pi) = (0, -1, \pi) \Rightarrow G \text{ is not a boundary.}$$

So, computing from the parametrization, we have

$$\int_G \vec{H} \cdot d\vec{x} = \int_0^\pi \vec{H}(\vec{x}(t)) \cdot \vec{x}'(t) dt$$

$$= \int_0^\pi \begin{pmatrix} \sin t \\ \sin t \\ \sin^2 t \cos t \end{pmatrix} \cdot \begin{pmatrix} \cos t \\ -\sin t \\ 1 \end{pmatrix} dt$$

$$= \int_0^\pi \sin t \cos t dt - \int_0^\pi \sin^2 t dt + \int_0^\pi \sin^2 t \cos t dt$$

$$= \left( \frac{1}{2} \sin^2 t \right) \Big|_0^\pi - \left( \frac{t - \frac{1}{2} \sin 2t}{2} \right) \Big|_0^\pi + \left( \frac{1}{3} \sin^3 t \right) \Big|_0^\pi$$

$$= 0 - \left( \frac{\pi}{2} - 0 \right) + 0$$

$$= \boxed{-\frac{\pi}{2}}$$

4. (20 pts) The region  $R$  consists of the points that are both above the cone  $z = \sqrt{x^2 + y^2}$  and below the plane  $z = 4$ . At a given moment, the vector field describing the flow of gnats in the vicinity of this region is

$$\vec{I}(x, y, z) = (3x - yz, xz + 2y, 8xz)$$

To address the question of the momentary rate of change of the total number of gnats in  $R$ , find the flux of  $\vec{I}$  through the inward oriented boundary of  $R$ .

$$\Phi = \iint_{\partial R} \vec{I} \cdot d\vec{S} = - \iint_{\partial R} \vec{I} \cdot d\vec{S} = - \iiint_R \nabla \cdot \vec{I} \, dV$$

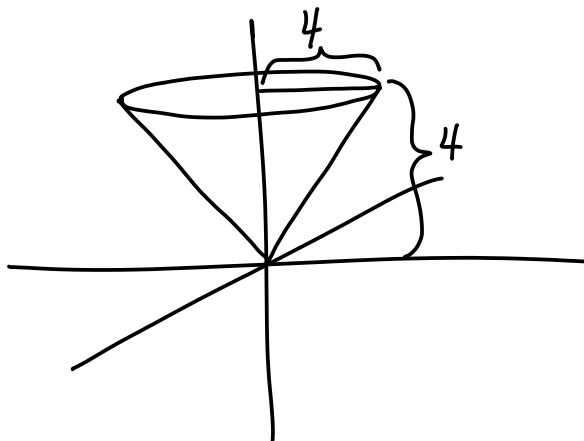
$$= - \iiint_R (5 + 8x) \, dV$$

$$= (-5) \iiint_R 1 \, dV + \iiint_R 8x \, dV$$

$$= (-5) (\text{volume of } R) + \bigcirc \quad \begin{array}{l} \text{by symmetry of } R \text{ through} \\ \text{the } yz\text{-plane.} \end{array}$$

$$= (-5) \left( \frac{1}{3} \pi (4)^2 (4) \right)$$

$$= \boxed{-\frac{320\pi}{3}}$$



5. (20 pts) The downward oriented surface  $P$  is the part of the paraboloid  $z = x^2 + y^2$  that is inside the cylinder  $(x-1)^2 + (y-1)^2 = 2$ . Compute the flux through  $P$  of the vector field  $\vec{W}$  below.

$$\vec{W}(x, y, z) = \begin{pmatrix} 3 - e^{z-2x-2y} \\ 4 + e^{z-2x-2y} \\ 14 \end{pmatrix} \quad \nabla \cdot \vec{W} = -e^{z-2x-2y}(-2) + e^{z-2x-2y}(-2)$$

$$+ 0$$

$$= 0 \Rightarrow \text{surf. ind.}$$

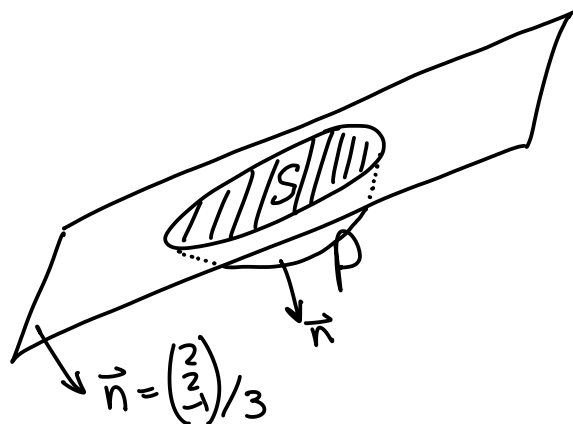
Intersection is where

$$(x-1)^2 + (y-1)^2 = 2 \quad \text{and} \quad z = x^2 + y^2$$

$$(x^2 - 2x + 1) + (y^2 - 2y + 1) = 2$$

$$(z) - 2x - 2y = 0$$

So the boundary of  $P$  is on this plane. So  $P$  has the same boundary as the surface  $S$  in the figure.



$$\text{So } \iint_P \vec{W} \cdot d\vec{S} = \iint_S \vec{W} \cdot d\vec{S}$$

$$= \iint_S \begin{pmatrix} 3 - e^{z-2x-2y} \\ 4 + e^{z-2x-2y} \\ 14 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} / 3 \, dS$$

$$= \iint_S 0 \, dS = \boxed{0}$$