EXAM 2

Math 212, 2017-2018 Fall, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.
All answers must be simplified. All of the policies and guidelines
on the class webpages are in effect on this exam.

Good luck!

Name Solutions

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ______________________

1. ____________

2. ____________

3. ____________

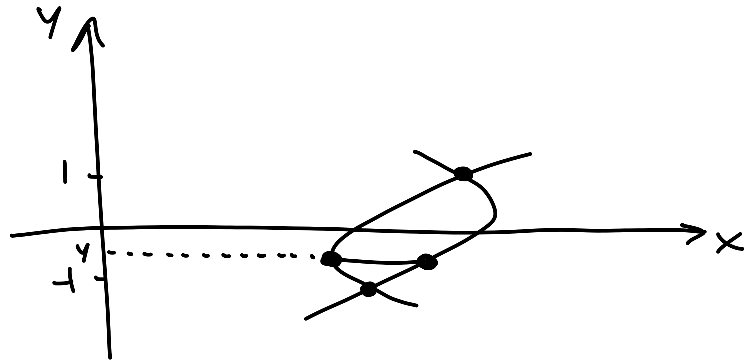
4. ____________

5. ____________

Total Score ____________ (/100 points)

1. *(20 pts)* The region R in $\mathbb{R}^2$ is bounded by the curves $x = y^2 + y + 5$ and $x = -y^2 + y + 7$. The distribution of mass on R has density $\delta(x, y) = y^2$.

(a) Compute the total mass on R.



Curves intersect when

$$y^2+y+5 = -y^2+y+7$$

$$2y^2-2=0$$

$$\Rightarrow y = \pm 1$$

So $$m = \iint \delta \, dA = \int_{-1}^{1} \int_{y^2+y+5}^{-y^2+y+7} y^2 \, dx \, dy$$

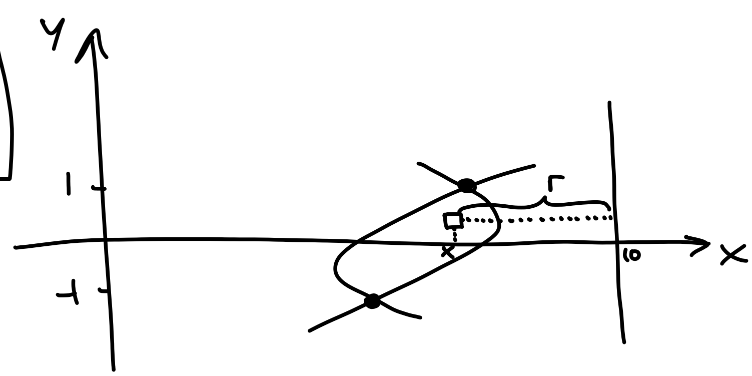
$$= \int_{-1}^{1} \left(xy^2 \right]_{x=y^2+y+5}^{x=-y^2+y+7} dy = \int_{-1}^{1} -2y^4 + 2y^2 \, dy$$

$$= -\frac{2}{5}y^5 + \frac{2}{3}y^3 \Big]_{-1}^{1} = 2\left(\frac{2}{3} - \frac{2}{5}\right) = \boxed{\frac{8}{15}}$$

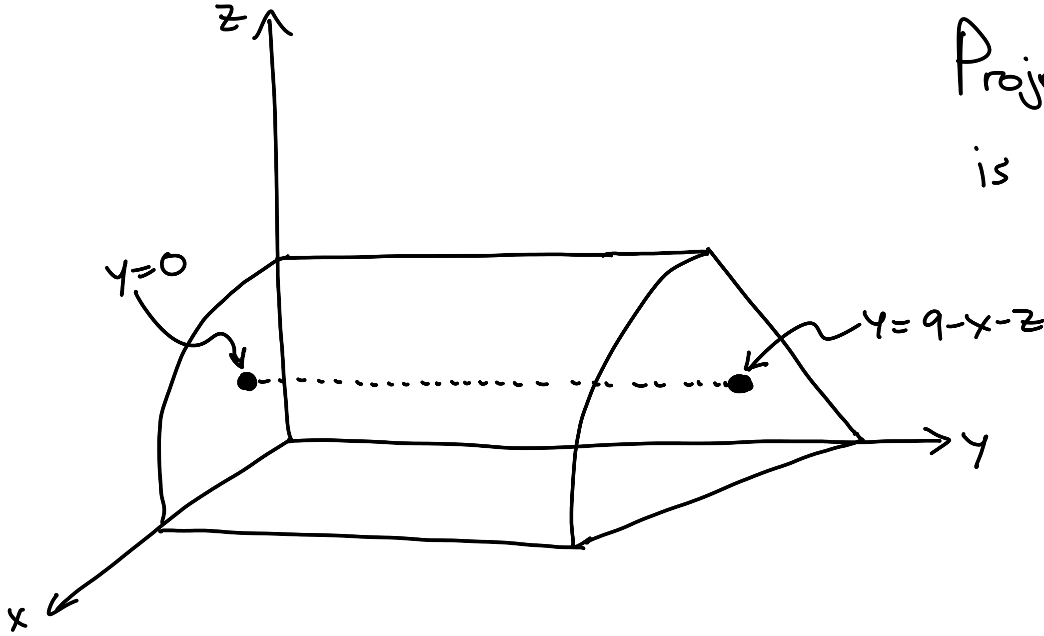
(b) Set up, but do not evaluate, a double iterated integral representing the moment of inertia of R around the line $x = 10$.

$$I = \iint r^2 \, dm = \iint (|x-10|)^2 \, \delta \, dA$$

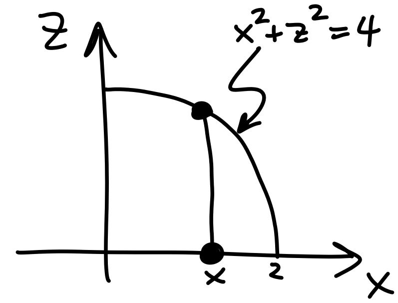
$$\boxed{= \int_{-1}^{1} \int_{y^2+y+5}^{-y^2+y+7} (10-x)^2 \, y^2 \, dx \, dy}$$



2. *(20 pts)* Set up, but do not evaluate, a triple iterated integral representing $\iiint_D f\, dV$, where $f(x, y, z) = 3z$ and D is the region in the first octant bounded by the coordinate planes and the surfaces $x^2 + z^2 = 4$ and $x + y + z = 9$.



Projection to xz-plane is

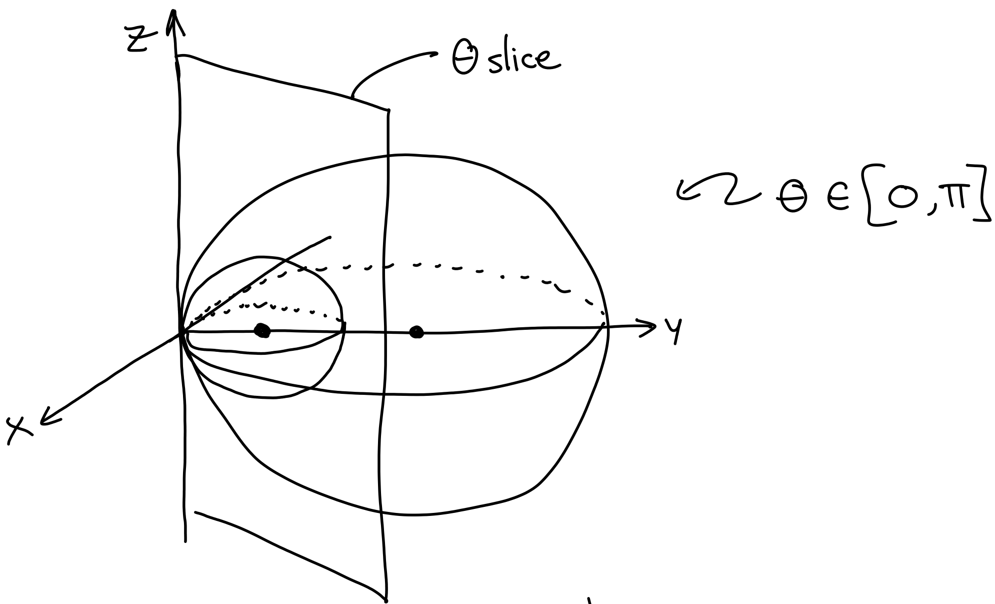


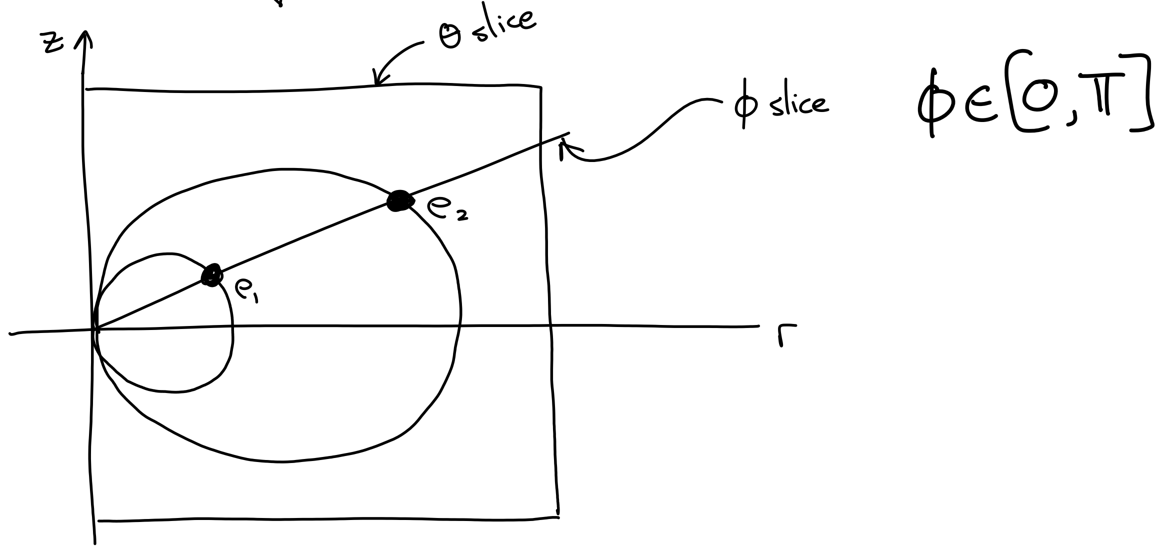
① $x \in [0,2]$

② $z \in [0, \sqrt{4-x^2}]$

So $\iiint_D f\, dV = \int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{9-x-z} f(x,y,z)\, dy\, dz\, dx$

3. *(20 pts)* The region T is defined by $x^2 + (y-1)^2 + z^2 \geq 1$ and $x^2 + (y-3)^2 + z^2 \leq 9$. Set up, but do not evaluate, a triple iterated integral in spherical coordinates representing the integral of $f(x, y, z) = x + y + z$ over T.





$x^2 + (y-1)^2 + z^2 = 1$

$x^2 + y^2 + z^2 - 2y = 0$

$\rho^2 - 2\rho \sin\phi \sin\theta = 0$

$\Rightarrow \rho_1 = 2\sin\phi\sin\theta$

$x^2 + (y-3)^2 + z^2 = 9$

$x^2 + y^2 + z^2 - 6y = 0$

$\rho^2 - 6\rho \sin\phi \sin\theta = 0$

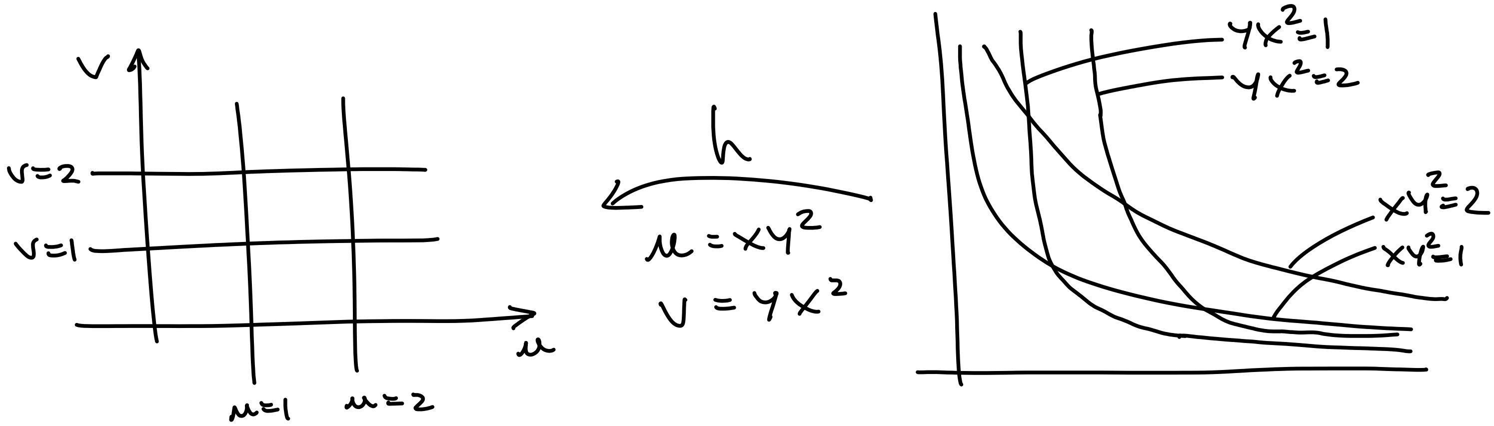
$\Rightarrow \rho_1 = 6\sin\phi\sin\theta$

Then $\iiint_T x+y+z \, dV$

$$= \int_0^{\pi}\int_0^{\pi}\int_{2\sin\phi\sin\theta}^{6\sin\phi\sin\theta} (\rho\sin\phi\cos\theta + \rho\sin\phi\sin\theta + \rho\cos\phi)\, \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

4. *(20 pts)*

(a) The domain C in the first quadrant of $\mathbb{R}^2$ is bounded by the curves $xy^2 = 1$, $xy^2 = 2$, $yx^2 = 1$, and $yx^2 = 2$. Use a change of variables to set up (but not evaluate) the integral of $f(x, y) = x^5y^5 \sin(xy^2 - yx^2)$ as an iterated integral.



$$\frac{\partial(u,v)}{\partial(x,y)} = \det\begin{pmatrix} y^2 & 2xy \\ 2xy & x^2 \end{pmatrix} = -3x^2y^2, \quad \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\frac{1}{-3x^2y^2}\right| = \frac{1}{3x^2y^2}$$

$$\iint f\,dx\,dy = \iint (f)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv = \int_1^2\int_1^2 \frac{x^5y^5}{3x^2y^2}\sin(xy^2 - yx^2)\,du\,dv$$

$$= \int_1^2\int_1^2 \frac{1}{3}\,uv\,\sin(u-v)\,du\,dv$$

(b) Find the value of the integral from part (a) without using change of variables.

Reflection over $y=x$ is by $R(x,y) = (y,x)$. The domain C is symmetric over this line. And

$$f(R(x,y)) = f(y,x) = y^5x^5\sin(yx^2 - xy^2)$$
$$= x^5y^5\sin(-(xy^2-yx^2)) = -x^5y^5\sin(xy^2-yx^2)$$
$$= -f(x,y)$$

so f has odd symmetry over this same line.

Therefore $\iint_C f\,dx\,dy = 0$ by symmetry.

5. *(20 pts)* The gradient vector for the function $f : \mathbb{R}^2 \to \mathbb{R}^1$ is

$$\nabla f(x, y) = (y^3 - 2xy, 3xy^2 - x^2)$$

The surface S is the part of the graph of f over the rectangle $[1, 2] \times [3, 4]$, and mass is distributed over S with density $\delta(x, y, z) = x^2y^2$.

Set up, but do not evaluate, a double iterated integral representing the mass on S.

$m = \iint_S \delta \, dS = \int_3^4 \int_1^2 \delta \sqrt{1 + f_x^2 + f_y^2} \, dx\,dy$ because S is the rectangular graph of f.

$$\nabla f = \begin{pmatrix} y^3 - 2xy \\ 3xy^2 - x^2 \end{pmatrix} = \begin{pmatrix} f_x \\ f_y \end{pmatrix}$$

so

$$m = \int_3^4 \int_1^2 \delta \sqrt{1 + f_x^2 + f_y^2} \, dx\,dy$$

$$= \int_3^4 \int_1^2 x^2y^2 \sqrt{1 + (y^3 - 2xy)^2 + (3xy^2 - x^2)^2} \, dx\,dy$$