## EXAM 1

Math 212, 2017-2018 Fall, Clark Bray.
You have 50 minutes.
No notes, no books, no calculators.
YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.
All answers must be simplified. All of the policies and guidelines on the class webpages are in effect on this exam.

Good luck!
Name Solutions

1. $\qquad$
2. $\qquad$
3. $\qquad$
4. $\qquad$
5. $\qquad$
6. $\qquad$
Total Score $\qquad$ (/100 points)
7. (18 pts) In this question we consider the vectors $\vec{v}=(3,1,0)$ and $\vec{w}=(1,0,2)$.
(a) Find the angle between these two vectors (leave your answer in terms of an inverse trig function).

$$
\theta=\arccos \frac{\vec{U} \cdot \vec{\omega}}{\|\vec{v}\|\|\vec{W}\|}=\arccos \frac{3}{\sqrt{10} \sqrt{5}}=\arccos \frac{3}{\sqrt{50}}
$$

(b) Find the component of $\vec{w}$ in the direction of $\vec{v}$.

$$
\operatorname{comp}_{\vec{v}}(\vec{w})=\frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\|}=\frac{3}{\sqrt{10}}
$$

(c) Find the area of the parallelogram $P$ defined by these two vectors.

$$
\begin{aligned}
& \vec{V} \times \vec{w}=\operatorname{det}\left(\begin{array}{ccc}
\vec{e}_{1} & \vec{e}_{2} & \vec{e}_{3} \\
3 & 1 & 0 \\
1 & 0 & 2
\end{array}\right)=(2,-6,-1) \\
& \text { area }=\|\vec{V} \times \vec{w}\|=\sqrt{(2)^{2}+(-6)^{2}+(-1)^{2}}=\sqrt{41}
\end{aligned}
$$

2. (15 pts) Find the equation of the plane $S$ that is parallel to $\vec{v}$ and perpendicular to the parallelogram $P$ (both from question 1 ), and goes through the point $(0,0,1)$.


$$
\begin{aligned}
& \vec{v} \times \vec{w}=(2,-6,-1) \\
& \begin{aligned}
\vec{n}=\vec{v} \times(\vec{v} \times \vec{w})= & \operatorname{det}\left(\begin{array}{ccc}
\vec{e}_{1} & \vec{e}_{2} & \vec{e}_{3} \\
3 & 1 & 0 \\
2 & -6 & -1
\end{array}\right) \\
& =(-1,3,-20)
\end{aligned} \\
& \vec{x}_{0}=(0,0,1)
\end{aligned}
$$

$$
\vec{n} \cdot \vec{x}=\vec{n} \cdot \vec{x}_{0} \Rightarrow-x+3 y-20 z=-20
$$

3. (18 pts) A particle with position $\vec{x}(t)$ has velocity $\vec{v}(t)=\left(2 t+1, e^{t}, \sin t\right)$ and $\vec{x}(0)=(1,2,3)$.
(a) Find an expression for $\vec{x}(t)$.

$$
\begin{aligned}
& \vec{X}(t)=\int \vec{v}(t) d t+\vec{c}=\int\left(\begin{array}{c}
2 t+\lambda \\
e^{2} \\
\sin t
\end{array}\right) d t+\vec{c} \\
& \vec{X}(t)=\left(\begin{array}{c}
t^{2}+t \\
e^{t} \\
-\cos t
\end{array}\right)+\vec{c}
\end{aligned}
$$

At $t=0$, we get

$$
\begin{aligned}
& \left(\begin{array}{l}
1 \\
2 \\
3
\end{array}\right)=\left(\begin{array}{c}
0 \\
1 \\
-1
\end{array}\right)+\vec{c} \quad \Rightarrow \vec{c}=\left(\begin{array}{l}
1 \\
1 \\
4
\end{array}\right) \\
& \vec{x}(t)=\left(\begin{array}{c}
x^{2}+t \\
e^{t} \\
-\cos t
\end{array}\right)+\left(\begin{array}{l}
1 \\
1 \\
4
\end{array}\right)
\end{aligned}
$$

(b) Let $L$ be the line tangent to the particle's path at time $t=0$. The symmetric equations for $L$ do not take the usual form due to a zero in the velocity vector; but the usual process of converting the parametrization does still yield the equations of a pair of planes defining $L$. Find those two equations.

At $t=0, \vec{v}=(1,1,0), \vec{x}=(1,2,3)$.
So the line is parametrized by

$$
\vec{x}=\left(\begin{array}{l}
1 \\
2 \\
3
\end{array}\right)+t\left(\begin{array}{l}
1 \\
1 \\
0
\end{array}\right) \quad \text { or } \quad \begin{aligned}
& x=1+1 t \\
& y=2+1 t \\
& z=3
\end{aligned}
$$

Solving for $t$ in the first two equations, we get

$$
x-1=y-2, z=3
$$

4. (18 pts) The surface $S$ has equation $x^{2} z-2 y^{4}+3 z=0$.
(a) Is $S$ the graph of a function $f: \mathbb{R}^{a} \rightarrow \mathbb{R}^{b}$ ? If not, why? And if so, find $a, b$, and an expression for $f$.

$$
z=\frac{2 y^{4}}{x^{2}+3}
$$

This is the graph $z=f(x, y)$ of the function $f\left(\mathbb{R}^{2} \rightarrow \mathbb{R}^{\prime}\right.$ defined by

$$
f(x, y)=\frac{2 y^{4}}{x^{2}+3}
$$

(b) Is $S$ a level set of a function $g: \mathbb{R}^{c} \rightarrow \mathbb{R}^{d}$ ? If not, why? And if so, find $c, d$, and an expression for $g$.

$$
x^{2} z-2 y^{4}+3 z=0
$$

This is the level set $g=0$ of the function $g: \mathbb{R}^{3} \rightarrow \mathbb{R}^{\prime}$ defined by

$$
g(x, y, z)=x^{2} z-2 y^{4}+3 z
$$

5. (16 pts) Find the equation of the tangent plane to the graph of the function $f(x, y)=3 x^{2}+5 y^{2}$ at the point $(2,1,17)$.

$$
\begin{array}{ll}
\frac{\partial f}{\partial x}=6 x & \left.\frac{\partial f}{\partial x}\right|_{(2,1)}=12 \\
\frac{\partial f}{\partial y}=10 y & \left.\frac{\partial f}{\partial y}\right|_{(2,1)}=10
\end{array}
$$

The linear approximation at $(2,1)$ then is

$$
L(x, y)=17+12(x-2)+10(y-1)
$$

and the tangent plane is the graph of $L$,

$$
z=17+12(x-2)+10(y-1)
$$

6. ( 15 pts) Due to the angle of the sun at a given moment, the shadow in the plane $x+2 y+5 z=0$ of a particle at position $(x, y, z)$ in space is given by

$$
\vec{s}(x, y, z)=\left(\begin{array}{c}
4 x+6 y+15 z \\
3 x+7 y+15 z \\
-2 x-4 y-9 z
\end{array}\right)
$$

At this moment, the particle is at $(3,2,1)$ and moving with velocity $(2,1,1)$. What is the velocity of this point's shadow?

$$
\dot{S}: \mathbb{R}^{3} \rightarrow \mathbb{R}^{3}
$$

By the chain rule, the input $(\vec{x})$ and output ( $\vec{s}$ ) velocities are related by

$$
\begin{aligned}
\frac{d \vec{s}}{d t} & =\left.J_{\vec{s}}\right|_{\vec{x}} \frac{d \vec{x}}{d t} \\
& =\left.\left(\begin{array}{lll}
\partial s_{1} / \partial x & \partial s_{1} / \partial y & \partial s_{1} / z_{z} \\
\partial s_{z} / x_{x} & \partial s_{3} / \partial_{y} & \partial s_{z} / z z \\
\partial s_{3} / \partial x & \partial s_{3} / \partial & \partial s_{3} / \partial z
\end{array}\right)\right|_{\vec{x}} \\
& =\left(\begin{array}{ccc}
4 & 6 & 15 \\
3 & 7 & 15 \\
-2 & 4 & -9
\end{array}\right)\left(\begin{array}{l}
2 \\
1 \\
1
\end{array}\right)=\left(\begin{array}{c}
29 \\
28 \\
-17
\end{array}\right)
\end{aligned}
$$

