EXAM 3

Math 212, 2016-2017 Spring, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.
All answers must be simplified. All of the policies and guidelines
on the class webpages are in effect on this exam.

Good luck!

Name Solutions

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ____________________

1. __________
2. __________
3. __________
4. __________
5. __________

Total Score __________ (/100 points)

1. *(22 pts)* The curve C is parametrized by $\vec{x}(t) = (\cos t, \sin t, t^2/\pi^2)$, with $t \in [0, 2\pi]$. The vector field $\vec{F}$ is given by $\vec{F}(x,y,z) = (\sin(yz), xz\cos(yz), xy\cos(yz))$.

Compute the line integral $\int_C \vec{F} \cdot d\vec{x}$.

$$\nabla \times \vec{F} = \begin{pmatrix} (x\cos yz - xyz\sin yz) - (x\cos yz - xyz\sin yz) \\ y\cos yz - y\cos yz \\ z\cos yz - z\cos yz \end{pmatrix} = \vec{0} \Rightarrow \vec{F} = \nabla f$$

(C is <u>not</u> closed, so it is <u>not</u> a boundary, so we <u>cannot</u> use Stokes's theorem.)

$$\left.\begin{aligned} f &= \int \sin yz \, dx = x\sin yz + c_1(y,z) \\ f &= \int xz\cos yz \, dy = x\sin yz + c_2(x,z) \\ f &= \int xy\cos yz \, dz = x\sin yz + c_3(y,z) \end{aligned}\right\} \Rightarrow f = x\sin yz$$

Then, by FTLI,

$$\begin{aligned} \int_C \vec{F} \cdot d\vec{x} = \int_C \nabla f \cdot d\vec{x} &= f(\vec{x}(2\pi)) - f(\vec{x}(0)) \\ &= f(1,0,4) - f(1,0,0) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

2. *(22 pts)* The path C consists of several line segments connecting vertices (in the order of the orientation on the curve, and intersecting only at the vertices) $(10,0,0)$, $(1,3,0)$, $(5,1,1)$, $(6,0,2)$, $(12,-2,2)$, $(16,-2,0)$, and then back to $(10,0,0)$. The vector field $\vec{F}$ is given by $\vec{F}(x,y,z) = (3x-y-z, 2x+y, x-3z)$.

Compute the line integral $\int_C \vec{F}\cdot d\vec{x}$. *(Hint: Note that all of the vertices of C are on the plane with equation $x+3y+2z=10$.)*

C is a closed curve in the plane $x+3y+2z=10$, so it is the boundary of a surface S contained in that plane.

So by Stokes's curl theorem,

$$\int_C \vec{F}\cdot d\vec{x} = \iint_S (\nabla\times\vec{F})\cdot\vec{n}\,dS$$

$$= \iint_S \begin{pmatrix}0\\-2\\3\end{pmatrix}\cdot\vec{n}\,dS$$

A normal vector to the plane is $\begin{pmatrix}1\\3\\2\end{pmatrix}$, so the unit normal vector is $\vec{n} = \begin{pmatrix}1\\3\\2\end{pmatrix}/\sqrt{14}$. Then

$$\int_C \vec{F}\cdot d\vec{x} = \iint_S \begin{pmatrix}0\\-2\\3\end{pmatrix}\cdot\begin{pmatrix}1\\3\\2\end{pmatrix}/\sqrt{14}\,dS$$

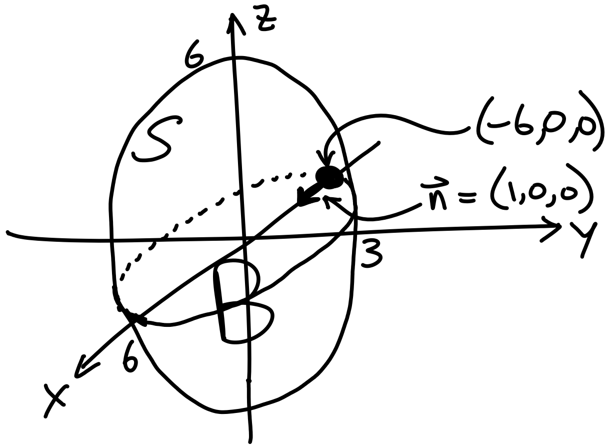
$$= \iint_S 0\,dS$$

$$= 0$$

3. *(22 pts)* The surface S is the ellipsoid with equation $x^2 + 4y^2 + z^2 = 36$, oriented such that $\vec{n} = (1, 0, 0)$ at the point $(-6, 0, 0)$.

The vector field $\vec{F}$ is given by $\vec{F}(x, y, z) = (x^2, \cos(x^3) + 3y, z^2 - 2xz)$.

Compute the surface integral $\iint_S \vec{F} \cdot \vec{n}\, dS$.



S is the boundary of the solid region B, but oriented inward instead of outward.

$$S = -\partial B$$

So, by Gauss's divergence theorem,

$$\iint_S \vec{F} \cdot \vec{n}\, dS = -\iint_{\partial B} \vec{F} \cdot \vec{n}\, dS = -\iiint_B \nabla \cdot \vec{F}\, dV$$

$$= -\iiint_B \left((2x) + (3) + (2z - 2x)\right) dV$$

$$= -\iiint_B 3 + 2z\, dV$$

$$= -\iiint_B 3\, dV - \underbrace{\iiint_B 2z\, dV}$$

$$= -3\,(\text{volume of } B)$$

$$= -3\left(\frac{4}{3}\pi (6)(3)(6)\right)$$

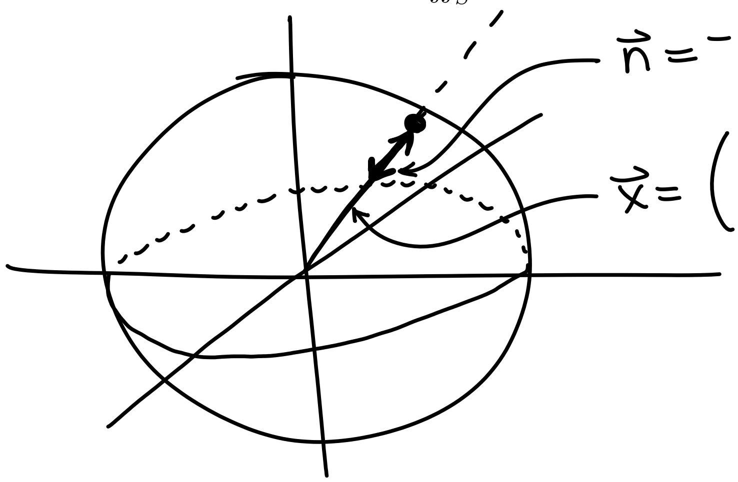
$$= -432\pi$$

$f = 2z$ is odd by reflection through the xy-plane ($R(x,y,z) = (x,y,-z)$, so $f(R(\vec{x})) = f(x,y,-z) = -2z = -f(\vec{x})$), and the region B is symmetric through the same plane. So this integral is zero.

4. *(12 pts)* The surface S is the sphere of radius 3 centered at the origin, oriented toward the origin.

The vector field $\vec{F}$ is given by $\vec{F}(x,y,z) = \left(\frac{ye^{x-z}}{1+y^2}, \frac{-xe^{x-z}}{1+y^2}, 0 \right)$.

Compute the surface integral $\iint_S \vec{F} \cdot \vec{n}\, dS$.



$$\vec{n} = -\frac{\vec{x}}{\|\vec{x}\|} = -\begin{pmatrix} x \\ y \\ z \end{pmatrix} / 3$$

$$\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

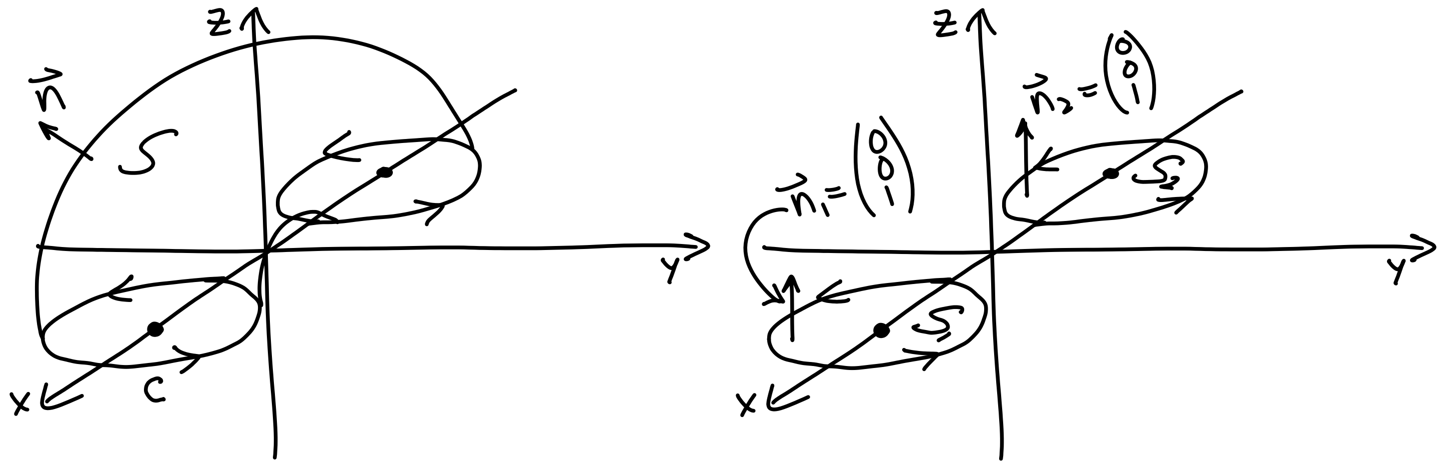
$$\vec{F} \cdot \vec{n} = \begin{pmatrix} \frac{ye^{x-z}}{1+y^2} \\ \frac{-xe^{x-z}}{1+y^2} \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -x \\ -y \\ -z \end{pmatrix} / 3 = 0$$

So $\iint_S \vec{F} \cdot \vec{n}\, dS = \iint_S 0\, dS = 0$

5. *(22 pts)* The curve C is the circle in the xy-plane of radius 2 centered at $(3,0,0)$. A torus (doughnut shaped surface) T is created by rotating C around the y-axis, and the surface S is the part of T with $z \geq 0$, oriented away from the "inside" of the doughnut.

The vector field $\vec{F}$ is given by $\vec{F}(x,y,z) = (2 - xy^2, 5 - x^2y, 3 + z(x^2+y^2))$.

Compute the surface integral $\iint_S \vec{F} \cdot \vec{n}\, dS$.



$$\nabla \cdot \vec{F} = \left((-y^2) + (-x^2) + (x^2+y^2)\right) = 0$$

So $\vec{F}$ is surface independent.

The surface S has the same boundary as the pair of surfaces S_1 and S_2 pictured above, so

$$\iint_S \vec{F}\cdot\vec{n}\,dS = \iint_{S_1} \vec{F}\cdot\vec{n}\,dS + \iint_{S_2} \vec{F}\cdot\vec{n}\,dS$$

$$= \iint_{S_1} \begin{pmatrix} 2-xy^2 \\ 5-x^2y \\ 3+z(x^2+y^2) \end{pmatrix} \cdot \begin{pmatrix}0\\0\\1\end{pmatrix} dS + \iint_{S_2} \begin{pmatrix} 2-xy^2 \\ 5-x^2y \\ 3+z(x^2+y^2) \end{pmatrix} \cdot \begin{pmatrix}0\\0\\1\end{pmatrix} dS$$

$$= \iint_{S_1} 3+z(x^2+y^2)\,dS + \iint_{S_2} 3+z(x^2+y^2)\,dS$$

$$= \iint_{S_1} 3\,dS + \iint_{S_2} 3\,dS \qquad \text{(because } z=0 \text{ on all of } S_1 \text{ and } S_2\text{).}$$

$$= 3\left(\pi(2)^2 + \pi(2)^2\right) = 24\pi$$