EXAM 1

Math 212, 2016-2017 Spring, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.
All answers must be simplified. All of the policies and guidelines
on the class webpages are in effect on this exam.

Good luck!

Name Solutions

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ____________________

1. __________

2. __________

3. __________

4. __________

5. __________

Total Score __________ (/100 points)

1. *(20 pts)*

(a) Find a vector that is perpendicular to both $(1, 2, 3)$ and $(2, 1, 1)$.

$$\begin{pmatrix}1\\2\\3\end{pmatrix} \times \begin{pmatrix}2\\1\\1\end{pmatrix} = \det\begin{pmatrix}\vec{e}_1 & \vec{e}_2 & \vec{e}_3\\ 1 & 2 & 3\\ 2 & 1 & 1\end{pmatrix} = \begin{pmatrix}-1\\5\\-3\end{pmatrix}$$

is perpendicular to both vectors.

(b) The component of $\vec{v} \times \vec{w}$ in the direction of the vector $\vec{x}$ is -2, and the volume of the parallelepiped defined by $\vec{v}, \vec{w}, \vec{x}$ is 6.

Compute the magnitude of $\vec{x}$, and decide if the listing $\vec{x}, \vec{v}, \vec{w}$ is in left hand order or right hand order. What about the listing $\vec{v}, \vec{x}, \vec{w}$?

① $\text{comp}_{\vec{x}}(\vec{v}\times\vec{w}) = \frac{\vec{x}}{\|\vec{x}\|}\cdot(\vec{v}\times\vec{w}) = -2$

② $\text{vol}(\|(\vec{x},\vec{v},\vec{w})) = |\vec{x}\cdot(\vec{v}\times\vec{w})| = 6$

$\Longrightarrow$

$\vec{x}\cdot(\vec{v}\times\vec{w}) = -6 \Longrightarrow \vec{x},\vec{v},\vec{w}$ is in LHO.

$\Downarrow$

$\|\vec{x}\| = 3$

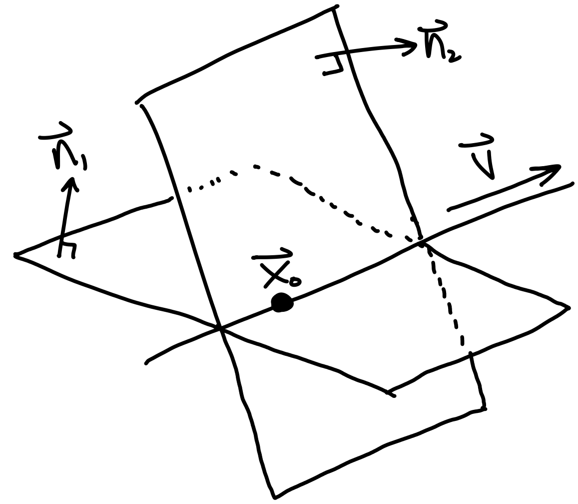
$\vec{v},\vec{x},\vec{w}$ is in RHO.

2. *(20 pts)* Find the symmetric equations for the line that is the intersection of the planes with equations below. Be sure to show all of the steps in the reasoning. *(Hint: Consider the possible relevance of the point* $(1, 1, 1)$*.)*

$$3x - 2y + z = 2 \quad \text{and} \quad x + 2y + z = 4$$

$(1,1,1)$ satisfies both of these equations, so it is on the intersection line. We can use this as our $\vec{x}_0$.

$\vec{v}$ must be parallel to both planes, and thus perpendicular to both normal vectors. So we choose



$$\vec{v} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -4 \\ -2 \\ 8 \end{pmatrix}$$

Then the line is parametrized by

$$\vec{x} = \vec{x}_0 + t\vec{v} \qquad \text{or} \qquad \begin{aligned} x &= 1 - 4t \\ y &= 1 - 2t \\ z &= 1 + 8t \end{aligned}$$

This could be true for a point (x, y, z) only if

$$t = \frac{x-1}{-4}$$
$$t = \frac{y-1}{-2}$$
$$t = \frac{z-1}{8}$$

So the symmetric equations are

$$\frac{x-1}{-4} = \frac{y-1}{-2} = \frac{z-1}{8}$$

3. *(20 pts)* A particle is moving with its velocity vector given by $\vec{v}(t) = (4t, 3t^2, e^t)$ and initial position $\vec{x}(0) = (1, 0, 2)$.

(a) Find an expression for the position of this particle as a function of t.

$$\vec{x}'(t) = \vec{v}(t) = (4t, 3t^2, e^t)$$

$$\vec{x}(t) = \int \vec{v}(t)\,dt = \int \begin{pmatrix} 4t \\ 3t^2 \\ e^t \end{pmatrix} dt = \begin{pmatrix} 2t^2 \\ t^3 \\ e^t \end{pmatrix} + \vec{c}$$

At $t=0$: $\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \vec{c} \Rightarrow \vec{c} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

So $\vec{x}(t) = \begin{pmatrix} 2t^2+1 \\ t^3 \\ e^t+1 \end{pmatrix}$

(b) Find the curvature of the particle's path at $t = 0$.

$$\vec{a}(t) = \vec{v}'(t) = \begin{pmatrix} 4t \\ 3t^2 \\ e^t \end{pmatrix}' = \begin{pmatrix} 4 \\ 6t \\ e^t \end{pmatrix}$$

Then $\vec{v}(0) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, $\vec{a}(0) = \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}$

$$\kappa(0) = \frac{\|\vec{v}(0) \times \vec{a}(0)\|}{v^3} = \frac{\|(0, 4, 0)\|}{1^3} = 4$$

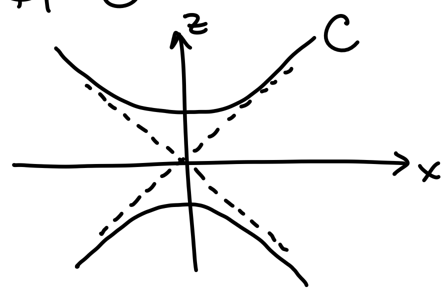
4. *(20 pts)* The surface S in $\mathbb{R}^3$ has equation $x^2 + y^2 - z^2 + 1 = 0$.

(a) Give a geometric description of S, and use a method from class to justify your description.

S is rotationally symmetric around the z-axis, and its cross section in the plane $y=0$ is the curve C with equation

$$\left.\begin{array}{c} x^2+y^2-z^2+1=0 \\ y=0 \end{array}\right\} \Rightarrow x^2-z^2+1=0$$

which is a hyperbola in the xz-plane:



So S is the hyperbola of two sheets obtained by rotating this curve C around the z-axis.

(b) Is S a level set of some function $f : \mathbb{R}^n \to \mathbb{R}^m$? If so, find n, m, and an expression for f.

Yes, S is the $f=0$ level set of the function $f: \mathbb{R}^3 \to \mathbb{R}^1$ given by

$$f(x,y,z) = x^2+y^2-z^2+1$$

(c) Is some portion of S a graph of some function $h : \mathbb{R}^a \to \mathbb{R}^b$? If so, describe which portion, and find a, b, and an expression for h.

Yes, consider the "top sheet" S_1 of S described by

$$x^2+y^2-z^2+1=0, \quad z>0$$

We can solve for z on S_1 by

$$z=\sqrt{x^2+y^2+1}$$

So, S_1 is the graph $z=h(x,y)$ of the function $h: \mathbb{R}^2 \to \mathbb{R}^1$ given by

$$h(x,y) = \sqrt{x^2+y^2+1}$$

5. *(20 pts)* Compute (or show that it does not exist) the limit below.

$$\lim_{(x,y)\to(0,0)} \frac{x^4 - y^6}{x^4 + y^4}$$

We consider straight line paths $y = mx$, on which the limit in question becomes

$$\lim_{x\to 0} \frac{x^4 - m^6x^6}{x^4 + m^4x^4}$$

$$= \lim_{x\to 0} \frac{1 - m^6x^2}{1 + m^4}$$

$$= \frac{1}{1+m^4}$$

This gives different results for different values of m.

So the original limit does not exist.