EXAM 3

Math 212, 2016-2017 Fall, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.
All answers must be simplified. All of the policies and guidelines
on the class webpages are in effect on this exam.

Good luck!

Name _____Solutions_____

"I have adhered to the Duke Community
Standard in completing this
examination."

1. _____

2. _____

Signature: _____

3. _____

4. _____

5. _____

Total Score _____ (/100 points)

1. *(20 pts)* Compute the value of the line integral

$$\int_C (e^{yz} + 2xz)\, dx + (xze^{yz})\, dy + (xye^{yz} + x^2)\, dz \quad = \int \underbrace{\begin{pmatrix} e^{yz} + 2xz \\ xze^{yz} \\ xye^{yz} + x^2 \end{pmatrix}}_{\vec{F}} \cdot d\vec{x}$$

where C is parametrized by $\vec{x}(t) = \left(t + t^2, \sin \pi t, \dfrac{6t}{t^2 + 1} \right)$, $t \in [0, 1]$.

$$\nabla \times \vec{F} = \vec{0}, \text{ so } \vec{F} = \nabla f$$

$$f = \int e^{yz} + 2xz \, dx = xe^{yz} + x^2 z + C_1(y, z) \qquad = x^2 z$$

$$f = \int xze^{yz} \, dy = xe^{yz} \qquad\qquad + \boxed{C_2(x, z)}$$

$$f = \int xye^{yz} + x^2 \, dz = xe^{yz} + x^2 z + C_3(x, y)$$

$$\text{So } f = xe^{yz} + x^2 z$$

By F.T.L.I.,

$$\int_C \vec{F} \cdot d\vec{x} = f(\vec{x}(1)) - f(\vec{x}(0))$$
$$= f(2, 0, 3) - f(0, 0, 0)$$
$$= 14$$

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2. *(20 pts)* The flow of dust through a region of space is described by the vector field

$$\vec{F} = (x - y + 2z)^2 \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$$

(a) The dust density is decreasing at both the point $A = (1, 2, 3)$ and the point $B = (3, 2, 1)$. What is the proportion between this rate at A and this rate at B?

$\frac{d\delta}{dt}$ is proportional to

$$\nabla \cdot \vec{F} = 2 \cdot 2(x - y + 2z)(1)$$
$$+ 3 \cdot 2(x - y + 2z)(-1)$$
$$+ 1 \cdot 2(x - y + 2z)(2) = 2(x - y + 2z)$$

$$\nabla \cdot \vec{F}(A) = 2(1 - 2 + 2 \cdot 3) = 10$$
$$\nabla \cdot \vec{F}(B) = 2(3 - 2 + 2 \cdot 1) = 6$$

So the proportion is $10/6 = 5/3$.

(b) Compute the flux of $\vec{F}$ through the boundary of the solid unit cube $R = [0,1] \times [0,1] \times [0,1]$.

$$\iint_{\partial R} \vec{F} \cdot d\vec{S} = \iiint_R \nabla \cdot \vec{F} \, dV$$
$$= \int_0^1 \int_0^1 \int_0^1 2(x - y + 2z) \, dx \, dy \, dz$$
$$= \int_0^1 \int_0^1 \left(x^2 - 2xy + 4xz \right]_{x=0}^{x=1} dy \, dz$$
$$= \int_0^1 \int_0^1 1 - 2y + 4z \, dy \, dz$$
$$= \int_0^1 \left(y - y^2 + 4yz \right]_{y=0}^{y=1} dz$$
$$= \int_0^1 4z \, dz$$
$$= 2$$

3. *(20 pts)* The surface S in $\mathbb{R}^3$ is upward oriented in the xy-plane, and consists of five unit squares arranged in a "+", with the first unit square being $[0, 1] \times [0, 1]$ and each of the other four unit squares sharing one of the edges with the first. The curve C is the boundary of S, consisting of 12 edges of length one.

The vector field $\vec{F}$ is given by

$$\vec{F} = \begin{pmatrix} 2x - y \\ y + 3z \\ 7x + 4z \end{pmatrix}$$

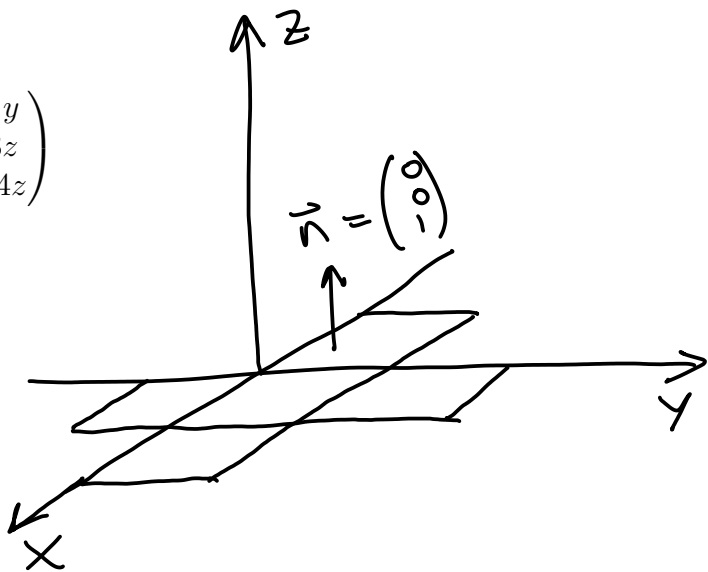Compute $\displaystyle\int_C \vec{F} \cdot d\vec{x}$.

by Stokes's thm

$$= \int_{\partial S} \vec{F} \cdot d\vec{x} = \iint_S (\nabla \times \vec{F}) \cdot \vec{n} \, dS$$

$$= \iint_S \begin{pmatrix} -3 \\ -7 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} dS$$
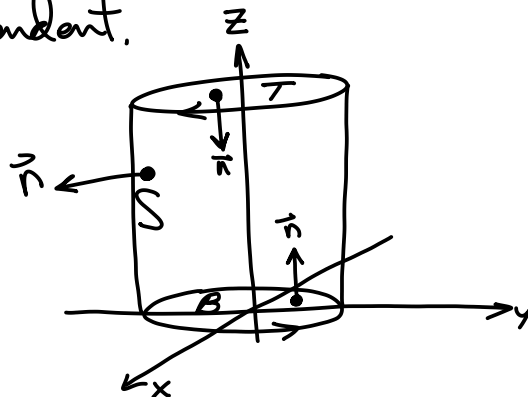
$$= \iint_S 1 \, dS$$

$$= \text{area of } S = 5$$

$$\vec{n} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

4. *(20 pts)* The surface S is the part of the cylinder $x^2 + y^2 = 1$ between $z = 0$ and $z = 4$, oriented away from the z-axis. The vector field $\vec{F}$ is given by

$$\vec{F} = \begin{pmatrix} x^2 + ye^y \\ e^{3x} - 2xy \\ 3 \end{pmatrix}$$

Compute $\iint_S \vec{F} \cdot d\vec{S}$.

$\nabla \cdot \vec{F} = 0$, so $\vec{F}$ is surface independent.

S has the same boundary curves as T+B, oriented down and up respectively.

Then

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_T \vec{F} \cdot d\vec{S} + \iint_B \vec{F} \cdot d\vec{S}$$

$$= \iint_T \vec{F} \cdot \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} dS + \iint_B \vec{F} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} dS$$

$$= \iint_T (-3) \, dS + \iint_B (3) \, dS$$

$$= -3 (\text{area of } T) + 3 (\text{area of } B)$$

$$= 0$$

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5. *(20 pts)* The vector field $\vec{F}$ is

$$\vec{F} = \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}$$

and the surface S is the part of the plane $3x + 9y + 3z = 24$ sitting above the square $[0, 1] \times [1, 2]$ in the xy-plane, and is oriented upward. Compute $\iint_S \vec{F} \cdot d\vec{S}$.

We can solve for z on S by $z = 8 - x - 3y$.

So we parametrize S with

$$\vec{x} = \begin{pmatrix} u \\ v \\ 8 - u - 3v \end{pmatrix} \qquad \begin{array}{l} u \in [0,1] \\ v \in [1,2] \end{array}$$

Then $\vec{x}_u = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, $\vec{x}_v = \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix}$, so $\vec{N} = \vec{x}_u \times \vec{x}_v$

$$= \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$$

this is consistent with the given upward orientation.

So $\iint_S \vec{F} \cdot d\vec{S} = \iint \vec{F} \cdot \vec{N} \, du \, dv$

$$= \iint \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} du \, dv$$

$$= \int_1^2 \int_0^1 10 \, du \, dv$$

$$= 10$$