EXAM 1

Math 212, 2016-2017 Fall, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.
All answers must be simplified. All of the policies and guidelines
on the class webpages are in effect on this exam.

Good luck!

Name Solutions

"I have adhered to the Duke Community
Standard in completing this
examination."

Signature: ______________________

1. ____________

2. ____________

3. ____________

4. ____________

5. ____________

Total Score ____________ (/100 points)

1. *(20 pts)* The cross product $\vec{a} \times \vec{b}$ is the vector $(2, 1, 3)$.

(a) Compute the area of the parallelogram defined by $\vec{a}$ and $\vec{b}$.

$$\text{area} = \|\vec{a} \times \vec{b}\| = \sqrt{2^2 + 1^2 + 3^2} = \sqrt{14}$$

(b) Compute the volume of the parallelepiped defined by $\vec{a}$, $\vec{b}$, and $\vec{c} = (1, -5, 2)$.

$$V = |\vec{c} \cdot (\vec{a} \times \vec{b})| = \left| \begin{pmatrix} 1 \\ -5 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \right| = 3$$

(c) Is the list $\vec{b}$, $\vec{c}$, $\vec{a}$ in right-hand order or left-hand order?

$$\vec{b} \cdot (\vec{c} \times \vec{a}) = \vec{c} \cdot (\vec{a} \times \vec{b}) = 3 > 0$$

So $\vec{b}, \vec{c}, \vec{a}$ is in R.H.O.

(d) Find the equation of the plane parallel to $\vec{a}$ and $\vec{b}$ that passes through the point $(1, 1, 1)$.

$\vec{a} \times \vec{b} = (2, 1, 3)$ is $\perp$ to plane, so set as $\vec{n}$.

$\vec{x}_0 = (1, 1, 1)$

$$\vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{x}_0$$

$$2x + y + 3z = 6$$

2. *(20 pts)* The velocity of the parametric curve $\vec{x}(t)$ is given by $\vec{v}(t) = (2\sin(3t), 4e^{2t})$, and a momentary position is $\vec{x}(0) = (4, 5)$.

(a) Find the position $\vec{x}(2)$.

$$\vec{x}(t) = \int \vec{v}(t)\,dt = \begin{pmatrix} -\frac{2}{3}\cos(3t) \\ 2e^{2t} \end{pmatrix} + \vec{C}$$

@ $t=0$: $$\begin{pmatrix} 4 \\ 5 \end{pmatrix} = \begin{pmatrix} -\frac{2}{3} \\ 2 \end{pmatrix} + \vec{C} \implies \vec{C} = \begin{pmatrix} 14/3 \\ 3 \end{pmatrix}$$

So $$\vec{x}(t) = \begin{pmatrix} -\frac{2}{3}\cos(3t) + 14/3 \\ 2e^{2t} + 3 \end{pmatrix}$$

$$\vec{x}(2) = \begin{pmatrix} -\frac{2}{3}\cos(6) + 14/3 \\ 2e^{4} + 3 \end{pmatrix}$$

(b) Find the acceleration and the curvature at time $t = 0$.

$$\vec{a}(t) = \vec{v}'(t) = \begin{pmatrix} 6\cos(3t) \\ 8e^{2t} \end{pmatrix}$$

$$\vec{v}(0) = \begin{pmatrix} 0 \\ 4 \end{pmatrix}, \quad \vec{a}(0) = \begin{pmatrix} 6 \\ 8 \end{pmatrix}$$

$$\kappa = \frac{|x'y'' - x''y'|}{v^3} = \frac{|0 \cdot 8 - 6 \cdot 4|}{4^3}$$

$$= \frac{24}{64} = \frac{3}{8}$$

3. *(20 pts)* The surface S has equation $x^3 - 2y^2 - 3z = 12$.

(a) Can S be viewed as a graph of some function $f : \mathbb{R}^a \to \mathbb{R}^b$? If so, identify a, b, and the formula for evaluating the function f.

$$\Longleftrightarrow \quad z = \frac{x^3 - 2y^2 - 12}{3}$$

This is the graph of $f : \mathbb{R}^{\overset{a}{2}} \to \mathbb{R}^{\overset{b}{1}}$ defined by

$$f(x,y) = \frac{x^3 - 2y^2 - 12}{3}$$

(b) Can S be viewed as a level set of some function $g : \mathbb{R}^c \to \mathbb{R}^d$? If so, identify c, d, and the formula for evaluating the function g.

This is the level set $g = 12$ of $g : \mathbb{R}^{\underset{c}{3}} \to \mathbb{R}^{\underset{d}{1}}$ defined by

$$g(x,y,z) = x^3 - 2y^2 - 3z$$

(c) Is the parametric curve $\vec{x}(t) = (-t^2, t^3, -t^6 - 4)$ contained in S?

With $x = -t^2$, $y = t^3$, $z = -t^6 - 4$, the equation for S is satisfied:

$$\begin{aligned} x^3 - 2y^2 - 3z &= (-t^2)^3 - 2(t^3)^2 - 3(-t^6 - 4) \\ &= -t^6 - 2t^6 + 3t^6 + 12 \\ &= 12 \ \checkmark \end{aligned}$$

So the curve is contained in S.

4. *(20 pts)* Does the limit below exist? If it does, compute the value; if it does not, show that it does not.

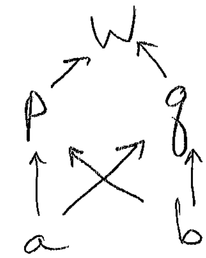
$$\lim_{(x,y)\to(0,0)} \frac{x^2y^3 - 5xy^4}{x^4 + 2x^2y^2 + y^4}$$

$$= \lim_{r\to 0} \frac{(r\cos\theta)^2 (r\sin\theta)^3 - 5(r\cos\theta)(r\sin\theta)^4}{r^4}$$

$$= \lim_{r\to 0} \underbrace{(r)}_{\text{approaches zero}} \underbrace{\left(\cos^2\theta\sin^3\theta - 5\cos\theta\sin^4\theta\right)}_{\text{bounded}}$$

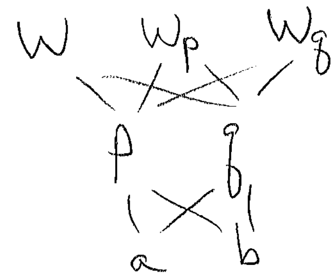
$$= 0$$

5. *(20 pts)* We have $w = w(p,q)$, $p = ab$, and $q = 2a + 3b$. Use the chain rule to write a fully simplified expression for $\dfrac{\partial^2 w}{\partial a\,\partial b}$.



$$\frac{\partial w}{\partial b} = \frac{\partial w}{\partial p}\frac{\partial p}{\partial b} + \frac{\partial w}{\partial q}\frac{\partial q}{\partial b}$$

$$= a w_p + 3 w_q$$



$$\frac{\partial}{\partial a}\left(\frac{\partial w}{\partial b}\right) = \frac{\partial}{\partial a}\left(a w_p + 3 w_q\right)$$

$$= \left(\frac{\partial}{\partial a}(a)\right) w_p + a\left(\frac{\partial}{\partial a}(w_p)\right) + 3\frac{\partial}{\partial a}(w_q)$$

$$= w_p + a\left(\frac{\partial w_p}{\partial p}\frac{\partial p}{\partial a} + \frac{\partial w_p}{\partial q}\frac{\partial q}{\partial a}\right) + 3\left(\frac{\partial w_q}{\partial p}\frac{\partial p}{\partial a} + \frac{\partial w_q}{\partial q}\frac{\partial q}{\partial a}\right)$$

$$= w_p + a\left(b w_{pp} + 2 w_{pq}\right) + 3\left(b w_{pq} + 2 w_{qq}\right)$$

$$= w_p + ab\, w_{pp} + q\, w_{pq} + 6 w_{qq}$$