EXAM 1

Math 212, 2015-2016 Spring, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.
All answers must be simplified. All of the policies and guidelines
on the class webpages are in effect on this exam.

Good luck!

Name Solutions

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ______________________

1. __________

2. __________

3. __________

4. __________

5. __________

Total Score __________ (/100 points)

1. *(20 pts)*

(a) Compute the area of the parallelogram with vertices at the points $(1,3,2)$, $(2,4,4)$, $(4,4,2)$, and $(5,5,4)$.

edge vectors are $\begin{pmatrix}2\\4\\4\end{pmatrix} - \begin{pmatrix}1\\3\\2\end{pmatrix} = \begin{pmatrix}1\\1\\2\end{pmatrix} = \vec{v}_1$

$\begin{pmatrix}4\\4\\2\end{pmatrix} - \begin{pmatrix}1\\3\\2\end{pmatrix} = \begin{pmatrix}3\\1\\0\end{pmatrix} = \vec{v}_2$

area $= \|\vec{v}_1 \times \vec{v}_2\| = \left\| \begin{pmatrix}-2\\6\\-2\end{pmatrix} \right\| = \sqrt{44}$

$= \boxed{2\sqrt{11}}$

(b) Show that if $\vec{v} \times \vec{w} \neq \vec{0}$, then the list $\vec{v}$, $\vec{w}$, $\vec{v} \times \vec{w}$ is in right hand order.

$\det\begin{pmatrix} \vec{v}\times\vec{w} \\ \vec{v} \\ \vec{w} \end{pmatrix} = (\vec{v}\times\vec{w}) \cdot (\vec{v} \times \vec{w})$ (triple product formula)

$= \|\vec{v}\times\vec{w}\|^2 > 0$ (b/c $\vec{v}\times\vec{w} \neq \vec{0}$)

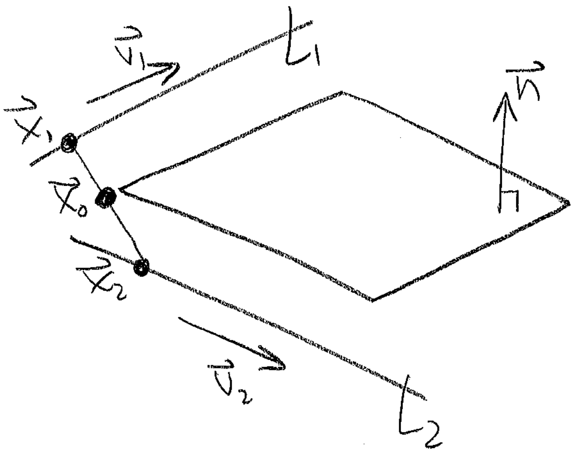
So $\vec{v}\times\vec{w}, \vec{v}, \vec{w}$ is in RHO, and then also

$\vec{v}, \vec{w}, \vec{v}\times\vec{w}$ is in RHO.

2. *(20 pts)* Find the equation of the plane that is parallel to and equidistant from the line L_1 parametrized by $\vec{x}(t) = (3+8t, 2+3t, 4t)$ and the line L_2 with symmetric equations

$$\frac{x-1}{4} = 4y+1 = z \quad = t \quad \Rightarrow \quad \begin{aligned} x &= 4t+1 \\ y &= \tfrac{1}{4}t - \tfrac{1}{4} \\ z &= t \end{aligned}$$

$$\vec{x}_1 = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}, \quad \vec{v}_1 = \begin{pmatrix} 8 \\ 3 \\ 4 \end{pmatrix} \qquad \vec{x}_2 = \begin{pmatrix} 1 \\ -1/4 \\ 0 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 4 \\ 1/4 \\ 1 \end{pmatrix}$$



Choose $\vec{n} = \vec{v}_1 \times \vec{v}_2$

$$= \begin{pmatrix} 2 \\ 8 \\ -10 \end{pmatrix}$$

Choose $\vec{x}_0 = \frac{1}{2}(\vec{x}_1 + \vec{x}_2)$

$$= \begin{pmatrix} 2 \\ 7/8 \\ 0 \end{pmatrix}$$

Then $\vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{x}_0$

$$\boxed{2x + 8y - 10z = 11}$$

3. *(20 pts)* A particle is moving such that its acceleration in terms of time t is

$$\vec{a}(t) = \begin{pmatrix} 4\sin 2t \\ 9e^{3t} \\ 6t \end{pmatrix}$$

and the initial position and initial velocity are given by $\vec{x}_0 = (1, 2, 3)$ and $\vec{v}_0 = (4, 5, 6)$, respectively.

(a) Find $\vec{x}(1)$.

$$\vec{v} = \int \vec{a}\,dt = \int \begin{pmatrix} 4\sin 2t \\ 9e^{3t} \\ 6t \end{pmatrix} dt = \begin{pmatrix} -2\cos 2t \\ 3e^{3t} \\ 3t^2 \end{pmatrix} + \vec{C}_1$$

$$\vec{v}_0 = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \\ 0 \end{pmatrix} + \vec{C}_1 \implies \vec{C}_1 = \begin{pmatrix} 6 \\ 2 \\ 6 \end{pmatrix}$$

$$\vec{x} = \int \vec{v}\,dt = \int \begin{pmatrix} -2\cos 2t + 6 \\ 3e^{3t} + 2 \\ 3t^2 + 6 \end{pmatrix} dt = \begin{pmatrix} -\sin 2t + 6t \\ e^{3t} + 2t \\ t^3 + 6t \end{pmatrix} + \vec{C}_2$$

$$\vec{x}_0 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \vec{C}_2 \implies \vec{C}_2 = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$$

$$\vec{x} = \begin{pmatrix} -\sin 2t + 6t + 1 \\ e^{3t} + 2t + 1 \\ t^3 + 6t + 3 \end{pmatrix} \qquad \vec{x}(1) = \begin{pmatrix} 7 - \sin(2) \\ e^3 + 3 \\ 10 \end{pmatrix}$$

(b) Find $\kappa(0)$. $\vec{v}(0) = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$ $\vec{a}(0) = \begin{pmatrix} 0 \\ 9 \\ 0 \end{pmatrix}$

$$\kappa = \frac{\|\vec{v} \times \vec{a}\|}{v^3} \qquad \vec{v} \times \vec{a} = \begin{pmatrix} -54 \\ 0 \\ 36 \end{pmatrix} \qquad v = \sqrt{77}$$

$$= \boxed{\frac{9\sqrt{52}}{77\sqrt{77}}}$$

4. *(20 pts)* The function $g : \mathbb{R}^3 \to \mathbb{R}$ is given by $g(x, y, z) = (x^2 + y^2 + z)^2$.

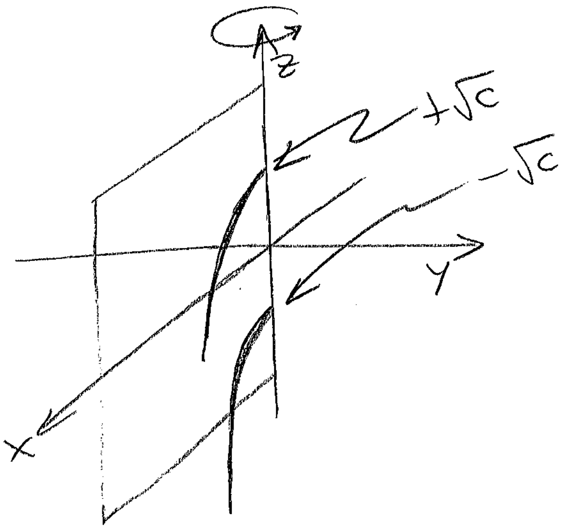
(a) Draw/describe a representative level set of g.

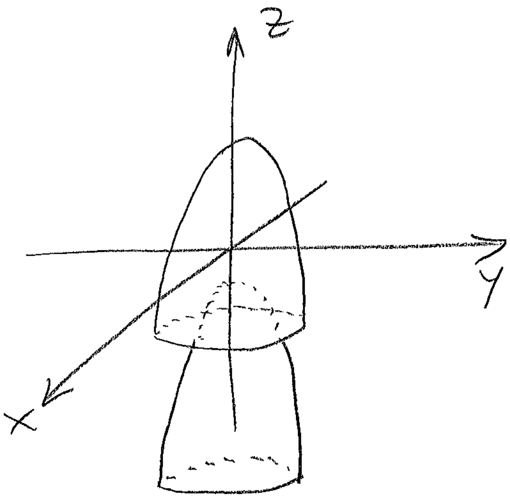
$$g = c \implies (x^2+y^2+z)^2 = c \implies x^2+y^2+z = \pm\sqrt{c}$$

when $c \geq 0$

$$\implies z = \pm\sqrt{c} - (x^2+y^2)$$

Each of these is rot. symm around z-axis, and parabolic in cross sections in xz-plane. So





(b) Find the unique function $f : \mathbb{R}^n \to \mathbb{R}^m$ whose graph is a level set of g, and indicate clearly the values of n and m.

The above level set fails the vertical line test unless $c = 0$. Then we have

$$z = -(x^2+y^2)$$

which is the graph of

$$f : \mathbb{R}^2 \to \mathbb{R}^1, \quad f(x,y) = -(x^2+y^2)$$

5. *(20 pts)*

(a) Find the linear approximation of the function given by $f(x, y, z) = \sin(\pi(xe^y + z))$ at the point $\vec{a} = (1, 0, 2)$.

$$f(\vec{a}) = \sin(3\pi) = 0$$

$$\nabla f = \begin{pmatrix} \cos(\pi(xe^y+z))(\pi e^y) \\ \cos(\pi(xe^y+z))(\pi x e^y) \\ \cos(\pi(xe^y+z))(\pi) \end{pmatrix} \qquad \nabla f(\vec{a}) = \begin{pmatrix} -\pi \\ -\pi \\ -\pi \end{pmatrix}$$

$$L(x,y,z) = f(\vec{a}) + \nabla f(\vec{a}) \cdot (\vec{x} - \vec{a})$$

$$= 0 + -\pi(x-1) - \pi(y-0) - \pi(z-2)$$

(b) Use the approximation above to estimate the value of $\sin(\pi(e^{0.02} + 2.03))$. $= f\begin{pmatrix} 1 \\ 0.02 \\ 2.03 \end{pmatrix}$

$$L\begin{pmatrix} 1 \\ 0.02 \\ 2.03 \end{pmatrix} = 0 - \pi(0) - \pi(0.02) - \pi(0.03)$$

$$= \frac{-\pi}{20}$$