EXAM 3
Math 212, 2015-2016 Fall, Clark Bray.
You have 50 minutes.
No notes, no books, no calculators.
YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.
All answers must be simplified. All of the policies and guidelines on the class webpages are in effect on this exam.
Good luck!
Name __________________________

“I have adhered to the Duke Community Standard in completing this examination.”

Signature: _________________________

1. ________

2. ________

3. ________

4. ________

5. ________

Total Score ________ (/100 points)
1. (20 pts) The surface $S$ is the part of the paraboloid $z = x^2 + y^2 - 4$ that is below the $xy$-plane, oriented downward. Compute the flux through $S$ of the vector field $\vec{F}(x, y, z) = (1+y, 2+z, 3+x)$.

\[ \nabla \cdot \vec{F} = 0, \text{ so } \vec{F} \text{ is surface independent.} \]

$S_2$, the disk of radius 2 in the $xy$-plane centered at $\hat{0}$, oriented downward, has the same boundary as $S$.

Thus,

\[ \iint_S \vec{F} \cdot d\vec{S} = \iint_{S_2} \vec{F} \cdot d\vec{S} = \iint_{S_2} \vec{F} \cdot (\hat{0}) dS \]

\[ = \iint_{S_2} (-3-x) dS \]

\[ = \iint_{S_2} (-3) dS \quad - \quad \iint_{S_2} \vec{F} \times d\vec{S} \]

\[ = -3 \text{(area)} \quad - \quad 0 \quad \text{by symmetry through y-axis} \]

\[ = (-3)(\pi(2)^2) \]

\[ = -12\pi \]
2. (20 pts) The surface $S$ is the boundary of the tetrahedron $T$ bounded by $x + 2y + 3z = 6$ and the coordinate planes. Compute the flux through $S$ of the vector field $\vec{F}(x, y, z) = (xz + y^2, yz + x^2, xy - z^2 + z)$.

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_T \nabla \cdot \vec{F} \, dV$$

$$\nabla \cdot \vec{F} = z + z - 2z + 1 = 1$$

$$\iiint_T (1) \, dV = \text{volume of } T$$

$$= \frac{1}{6} (6)(3)(3) = 6$$
3. (20 pts) The curve $C$ is parametrized by $\vec{r}(t) = (t^2 + 1, 2t - 3t^2, t^3)$, starting at $t = 0$ and ending at $t = 1$. Compute the line integral along $C$ of the vector field $\vec{F}(x, y, z) = (ye^{xy} + z, xe^{xy}, x)$.

\[
\nabla \times \vec{F} = \begin{pmatrix} 0 & -\frac{1}{y} & \frac{1}{y^2} \\ \frac{1}{y^2} & 0 & -1 \\ -\frac{1}{y} & \frac{x}{y^2} & 0 \end{pmatrix} = \vec{0}, \quad \text{so} \quad \vec{F} = \nabla f
\]

\[f = e^{xy} + xz + C_1(y/z)\]

\[f = e^{xy} + C_2(x/z)\]

\[f = xz + C_3(xy)\]

\[\Rightarrow \text{can use} \quad f = e^{xy} + xz\]

\[
\int_C \vec{F} \cdot d\vec{r} = f(\vec{r}(1)) - f(\vec{r}(0))
\]

\[= f(2, -1, 1) - f(1, 0, 0)\]

\[= \left( e^{-2} + 2 \right) - \left( e^0 + 0 \right)\]

\[= \frac{e^{-2} + 1}{e^0 + 1}\]

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4. (20 pts) The curve $C$ is the unit circle in the $yz$-plane, oriented clockwise as seen from the positive part of the $x$-axis. Compute the line integral along $C$ of the vector field $\vec{F}(x, y, z) = (yz + z, xz + x, xy + y)$.

\[
\nabla \times \vec{F} = \begin{pmatrix}
x + 1 & -x \\
y + 1 & -y \\
z + 1 & -z
\end{pmatrix} = \begin{pmatrix} 1 \\
1 \\
1
\end{pmatrix}
\]

$C = \partial S$, where $S$ is the unit disk in $yz$-plane, with $\vec{n} = (-1, 0, 0)$.

Then

\[
\int_C \vec{F} \cdot d\vec{x} = \iint_S (\nabla \times \vec{F}) \cdot \vec{n} \, dS
\]

\[
= \iint_S \begin{pmatrix} 1 \\
1 \\
1
\end{pmatrix} \cdot \begin{pmatrix} -1 \\
0 \\
0
\end{pmatrix} \, dS
\]

\[
= \iint_S (-1) \, dS
\]

\[
= -\text{area of } S
\]

\[
= \sqrt{-\pi}
\]
5. (20 pts) The surface $S_1$ is the part of the cylinder $x^2 + y^2 = 1$ with $y \geq 0$, and the surface $S_2$ is the plane $y = z$; the curve $C$ is the intersection of these two surfaces, oriented in the negative $x$-direction. Compute the line integral along $C$ of the vector field $\vec{F}(x, y, z) = (y, z, x)$.

\[ \nabla \times \vec{F} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} \neq 0 \]

So cannot use F.T.I.I.,

$C$ is not a boundary

So cannot use Stokes.

Parametrize $C$, noting

that $x, y$ follow unit circle in $xy$-plane

$(x = \cos t, \; y = \sin t)$, and $z = y$ (so $z = \sin t$),

So

$\vec{x} = (\cos t, \; \sin t, \; \sin t) \quad t \in [0, \pi]$

$\vec{x}' = (-\sin t, \; \cos t, \; \cos t)$

Then

\[ \int_C \vec{F} \cdot d\vec{r} = \int_0^\pi \vec{F} \cdot \vec{x}' \; dt = \int_0^\pi \left( \frac{y}{x} \right) - \left( \frac{-\sin t}{\cos t} \right) \; dt \]

\[ = \int_0^\pi \left( \frac{\sin t}{\cos t} \right) \cdot \left( \frac{-\sin t}{\cos t} \right) \; dt \]

\[ = \int_0^\pi \left( -\sin^2 t \right) dt + \int_0^\pi \sin t \cos t dt + \int_0^\pi \cos^2 t dt \]

\[ = \int_0^\pi -\frac{1 - \cos 2t}{2} dt + \int_0^\pi \frac{\sin 2t}{2} dt + \int_0^\pi \frac{1 + \cos 2t}{2} dt \]

\[ = \frac{-\pi}{2} + 0 + \frac{\pi}{2} = 0 \]