EXAM 2

Math 212, 2015-2016 Fall, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.
All answers must be simplified. All of the policies and guidelines
on the class webpages are in effect on this exam.

Good luck!

Name Solutions

"I have adhered to the Duke Community
Standard in completing this
examination."

Signature: ____________________

1. __________

2. __________

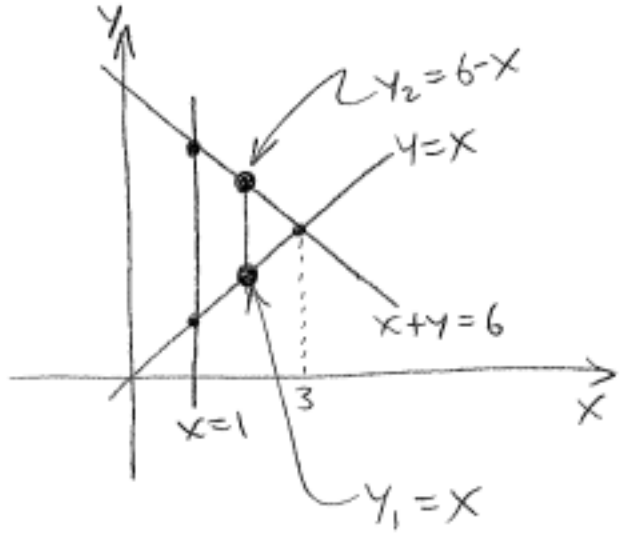
3. __________

4. __________

5. __________

Total Score __________ (/100 points)

1. *(20 pts)* The triangular region D in the xy-plane has vertices at $(1,1)$, $(1,5)$, and $(3,3)$. Bacteria are spread over this region with population density given by $\delta(x,y) = 1000x$. Compute the total population of bacteria in this region.



$$\text{pop.} = \iint_D \delta \, dA$$

$$= \iint_D 1000x \, dA$$

$$= 1000 \int_1^3 \int_x^{6-x} x \, dy \, dx$$

$$= 1000 \int_1^3 \left(x(6-x) - x(x) \right) dx$$

$$= 1000 \int_1^3 6x - 2x^2 \, dx$$

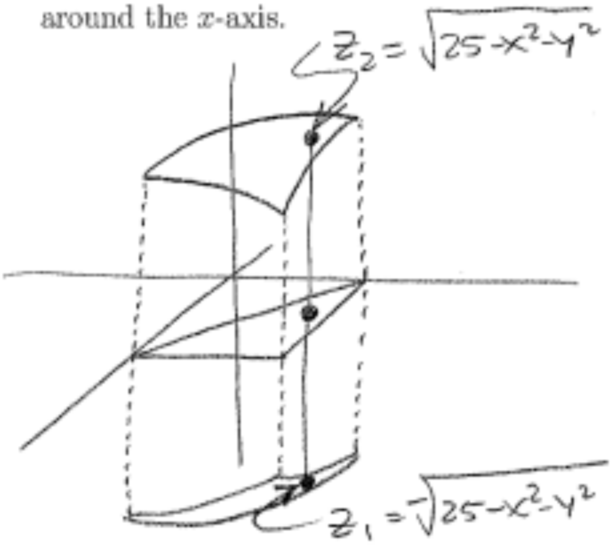
$$= 1000 \left(3x^2 - \frac{2}{3}x^3 \right]_1^3$$

$$= 1000 \left((27-18) - \left(3 - \frac{2}{3}\right) \right)$$

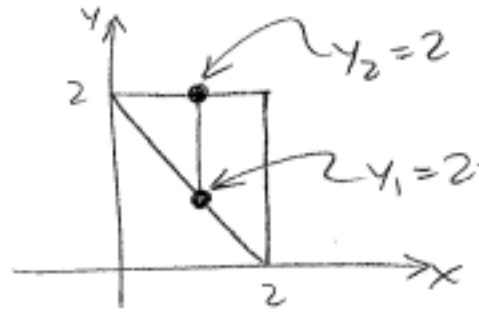
$$= \frac{20{,}000}{3}$$

2. *(20 pts)* The region R is inside the prism bounded by $x = 2$, $y = 2$, $x + y = 2$, and inside the sphere centered at the origin of radius 5. The density in R is given by $\delta = e^z$.

Set up, but do not evaluate, a triple iterated integral representing the moment of inertia of R around the x-axis.



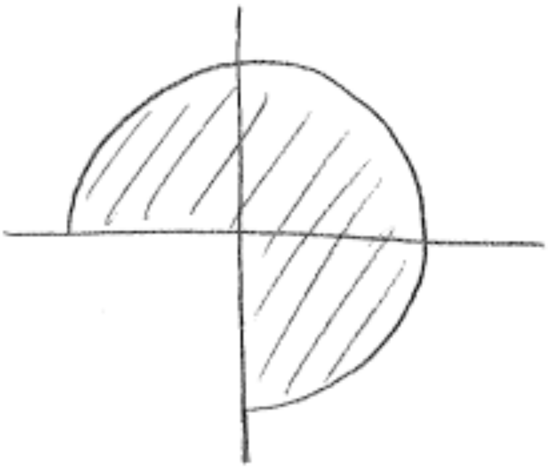
Projection to xy-plane is:



$$I = \iiint_R r^2 \delta \, dV = \iiint (y^2+z^2)(e^z)\, dV$$

$$= \int_0^2 \int_{2-x}^{2} \int_{-\sqrt{25-x^2-y^2}}^{\sqrt{25-x^2-y^2}} (y^2+z^2)(e^z)\, dz\, dy\, dx$$

3. *(20 pts)* The region D in the xy-plane is the part of the unit disk excluding the third quadrant. Compute the integral over this region of the function $f(x,y) = e^{x^2+y^2}$.



$$\iint_D e^{x^2+y^2}\,dx\,dy$$

$$= \iint e^{r^2}\, r\, dr\, d\theta$$

$$= \int_{-\pi/2}^{\pi} \int_0^1 e^{r^2}\, r\, dr\, d\theta$$

$$= \int_{-\pi/2}^{\pi} \left(\tfrac{1}{2} e^{r^2}\right]_0^1 d\theta$$

$$= \int_{-\pi/2}^{\pi} \tfrac{1}{2}(e-1)\, d\theta$$

$$= \frac{3\pi(e-1)}{4}$$

4. *(20 pts)* Use a parametrization to compute the area of the sphere of radius R centered at the origin.

$$\rho = R \text{ gives } \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} R\sin\phi\cos\theta \\ R\sin\phi\sin\theta \\ R\cos\phi \end{pmatrix}$$

$$\vec{X}_\phi = \begin{pmatrix} R\cos\phi\cos\theta \\ R\cos\phi\sin\theta \\ -R\sin\phi \end{pmatrix} \qquad \vec{X}_\theta = \begin{pmatrix} -R\sin\phi\sin\theta \\ R\sin\phi\cos\theta \\ 0 \end{pmatrix}$$

$$\vec{X}_\phi \times \vec{X}_\theta = \begin{pmatrix} R^2\sin^2\phi\cos\theta \\ R^2\sin^2\phi\sin\theta \\ R^2\cos\phi\sin\phi(\cos^2\theta + \sin^2\theta) \end{pmatrix} = \begin{pmatrix} R^2\sin^2\phi\cos\theta \\ R^2\sin^2\phi\sin\theta \\ R^2\cos\phi\sin\phi \end{pmatrix}$$

$$\|\vec{N}\| = \sqrt{(R^2\sin^2\phi\cos\theta)^2 + (R^2\sin^2\phi\sin\theta)^2 + (R^2\cos\phi\sin\phi)^2}$$

$$= \sqrt{R^4\sin^4\phi + R^4\cos^2\phi\sin^2\phi}$$

$$= \sqrt{R^4\sin^2\phi(\sin^2\phi + \cos^2\phi)} = R^2\sin\phi$$

$$\text{area} = \int_0^{2\pi}\int_0^{\pi} R^2\sin\phi \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \left(-R^2\cos\phi\right]_0^{\pi} d\theta$$

$$= \int_0^{2\pi} 2R^2 \, d\theta = 4\pi R^2$$

5. *(20 pts)* Compute $\int_C \vec{F} \cdot d\vec{x}$, where C is the portion of $y = x^2$ starting at $(0,0)$ and ending at $(2,4)$, and $\vec{F}(x,y) = (y,3)$.

$$\vec{x}(t) = \begin{pmatrix} t \\ t^2 \end{pmatrix} \quad , \quad t \in [0,2]$$

$$\vec{x}'(t) = \begin{pmatrix} 1 \\ 2t \end{pmatrix}$$

$$\int_c \vec{F} \cdot d\vec{x} = \int_0^2 \vec{F} \cdot \vec{x}' \, dt$$

$$= \int_0^2 \begin{pmatrix} t^2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2t \end{pmatrix} dt$$

$$= \int_0^2 t^2 + 6t \, dt$$

$$= \left(\tfrac{1}{3} t^3 + 3t^2 \right]_0^2$$

$$= \frac{44}{3}$$