EXAM 1

Math 212, 2015-2016 Fall, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.
All answers must be simplified. All of the policies and guidelines
on the class webpages are in effect on this exam.

Good luck!

Name Solutions

Disc.: Number________ TA______________ Day/Time______________

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ________________

1. ________

2. ________

3. ________

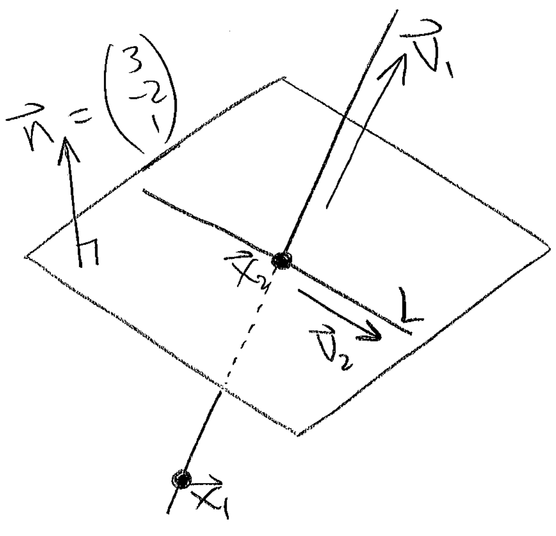
4. ________

5. ________

6. ________

Total Score ________ (/100 points)

1. *(20 pts)* Find a parametrization of the line L that is contained in the plane with equation $3x-2y+z=4$ and perpendicular to the line parametrized by $(x,y,z)=(2+t,3-2t,1+t)$.



$$\vec{x} = \underbrace{\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}}_{\vec{x}_1} + t \underbrace{\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}}_{\vec{v}_1}$$

$\vec{v}_2 \perp \vec{v}_1, \vec{n}$, so we choose

$$\vec{v}_2 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix}$$

To find $\vec{x}_2$ we solve for t generating a point satisfying the equation of a plane:

$$3(2+t) - 2(3-2t) + (1+t) = 4$$

$$8t + 1 = 4 \Rightarrow t = \frac{3}{8}$$

$$\Rightarrow \vec{x}_2 = \begin{pmatrix} 19/8 \\ 9/4 \\ 11/8 \end{pmatrix}$$

2. *(10 pts)* The curve C is parametrized by

$$\vec{x}(t) = \begin{pmatrix} t-1 \\ t^2 - t \\ t^3 - t^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow t = 1$$

At the moment this curve passes through the origin, compute the velocity, acceleration, and curvature.

$$\vec{v}(t) = \vec{x}'(t) = \begin{pmatrix} 1 \\ 2t-1 \\ 3t^2-2t \end{pmatrix} \quad \text{so} \quad \vec{v}(1) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\vec{a}(t) = \vec{x}''(t) = \begin{pmatrix} 0 \\ 2 \\ 6t-2 \end{pmatrix} \quad \text{so} \quad \vec{a}(1) = \begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix}$$

$$\kappa(1) = \frac{\|\vec{v}(1) \times \vec{a}(1)\|}{(v(1))^3} = \frac{\|(2,-4,2)\|}{(\sqrt{3})^3} = \frac{\sqrt{24}}{3\sqrt{3}}$$

$$= \frac{2\sqrt{2}}{3}$$

3. *(10 pts)* Compute the limit below, or show that it does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{x^4 - 3xy^5}{x^4 + y^4}$$

Along $y = mx$:

$$\lim_{x\to 0} \frac{x^4 - 3m^5x^6}{x^4 + m^4x^4} = \lim_{x \to 0} \frac{1 - 3m^5x^2}{1 + m^4} = \frac{1}{1+m^4}$$

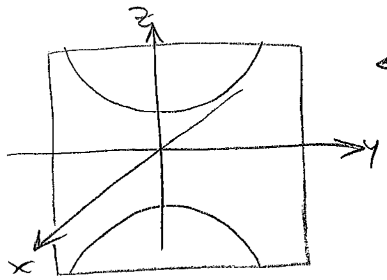
This has different values along different lines, so the limit D.N.E.

4. *(20 pts)* The surface Q has equation $-x^2 - y^2 + z^2 = 1$; the surface S is the part of Q above the xy-plane.

(a) Is Q a circular paraboloid, elliptical paraboloid, ellipsoid, hyperbolic paraboloid, hyperboloid of 1 sheet, hyperboloid of 2 sheets, hyperbolic cylinder, parabolic cylinder, or none of these? Explain your reasoning (do not simply cite a result from the book!).

Q is rotationally symmetric around the z-axis.

In yz-plane ($x=0$) we have $z^2-y^2=1$, which is this hyperbola. Q is the <u>hyperboloid of two sheets</u> that results from rotating this around the z-axis



(b) Is Q a level set of a function f? If so, indicate the domain, target, and formula for f.

$-x^2-y^2+z^2=1 \iff f(x,y,z)=1$, where

$f:\mathbb{R}^3 \to \mathbb{R}^1$ is given by

$f(x,y,z) = -x^2-y^2+z^2$

So yes, Q is a level set.

(c) Is S the graph of a function g? If so, indicate the domain, target, and formula for g.

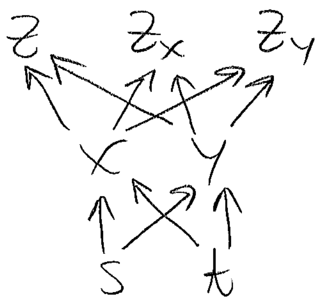
$\underbrace{-x^2-y^2+z^2=1, \quad z \geq 0}_{\text{def. of } S} \iff z = +\sqrt{1+x^2+z^2}$

This is the graph of

$g: \mathbb{R}^2 \to \mathbb{R}^1$ given by

$g(x,y) = \sqrt{1+x^2+z^2}$

5. *(20 pts)* Suppose that z is a function of x and y, which themselves are functions of s and t as given by $x = 3s^2$ and $y = 2s - 4t^2$. Find a fully simplified expression for $\dfrac{\partial^2 z}{\partial s^2}$.



$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$= (z_x)(6s) + (z_y)(2)$$

$$\frac{\partial^2 z}{\partial s^2} = \frac{\partial}{\partial s}\left(\frac{\partial z}{\partial s}\right) = \frac{\partial}{\partial s}\Big((z_x)(6s) + 2z_y\Big)$$

$$= \left[\frac{\partial z_x}{\partial s}\right](6s) + (z_x)(6) + 2\left[\frac{\partial z_y}{\partial s}\right]$$

$$= \left[\frac{\partial z_x}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z_x}{\partial y}\frac{\partial y}{\partial s}\right](6s) + 6z_x + 2\left[\frac{\partial z_y}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z_y}{\partial y}\frac{\partial y}{\partial s}\right]$$

$$= \left[(z_{xx})(6s) + (z_{xy})(2)\right](6s) + 6z_x + 2\left[(z_{xy})(6s) + (z_{yy})(2)\right]$$

$$= 6z_x + 36s^2 z_{xx} + 24s\, z_{xy} + 4z_{yy}$$

6. *(20 pts)* The concentration of air pollution (in units of ppm) in a region is given by

$$C(x, y) = \arctan(6 - x^2 - y^2)$$

Use the directional derivative to compute $\frac{dC}{ds}$ for a particle that is at the point $(1, 2)$ and moving with velocity $(5, 12)$.

$$\nabla C = \begin{pmatrix} \partial C/\partial x \\ \partial C/\partial y \end{pmatrix} = \begin{pmatrix} \frac{1}{1+(6-x^2-y^2)^2} \cdot (-2x) \\ \frac{1}{1+(6-x^2-y^2)^2} \cdot (-2y) \end{pmatrix}$$

$$\nabla C(1,2) = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$$

$$\frac{dC}{ds} = D_{\vec{u}} C(1,2) = \nabla C(1,2) \cdot \vec{u}$$

$$\vec{u} = \frac{(5,12)}{\|(5,12)\|} = \frac{(5,12)}{13}$$

$$= \begin{pmatrix} -1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 5/13 \\ 12/13 \end{pmatrix} = \frac{-29}{13}$$