EXAM 3

Math 212, 2014-2015 Spring, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.
All answers must be simplified. All of the policies and guidelines
on the class webpages are in effect on this exam.

Good luck!

Name Solutions

Disc.: Number________ TA______________ Day/Time______________

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ________________

1. ________

2. ________

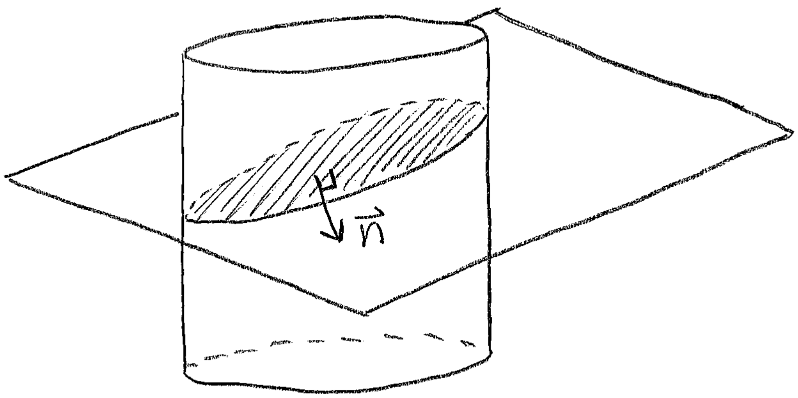
3. ________

4. ________

5. ________

Total Score ________ (/100 points)

1. *(20 pts)* The downward oriented surface S is the part of the plane $2x - 3y + 5z = 0$ that is inside the cylinder with equation $x^2 + y^2 = 1$. Compute the flux through this surface of the vector field $\vec{G}(x, y, z) = \langle 2x, x - y, y + z \rangle$.



$$2x - 3y + 5z = 0$$

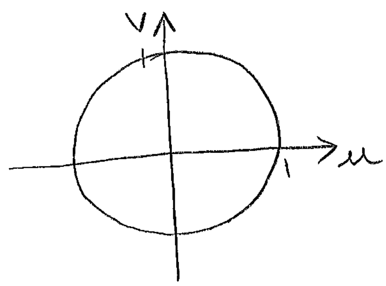
$$z = \frac{3y - 2x}{5}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} u \\ v \\ (3v - 2u)/5 \end{pmatrix}$$

$$\vec{N} = \vec{X}_u \times \vec{X}_v = \begin{pmatrix} 1 \\ 0 \\ -2/5 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 3/5 \end{pmatrix} = \begin{pmatrix} 2/5 \\ -3/5 \\ 1 \end{pmatrix}$$

oriented incorrectly, so

$$\Phi = -\iint_D \vec{G} \cdot \vec{N} \, du\,dv$$



$$= -\iint_D \begin{pmatrix} 2x \\ x - y \\ y + z \end{pmatrix} \cdot \begin{pmatrix} 2/5 \\ -3/5 \\ 1 \end{pmatrix} du\,dv$$

$$= \iint_D \frac{-x - 8y - 5z}{5} \, du\,dv$$

$$= \iint_D \frac{-u - 8v - (3v - 2u)}{5} \, du\,dv$$

$$= \iint_D \frac{u - 11v}{5} \, du\,dv$$

$$= \underbrace{\iint_D \frac{u}{5} \, du\,dv}_{\substack{D \text{ is symm over } v\text{-axis} \\ \frac{u}{5} \text{ has odd symm}}} + \underbrace{\iint_D \frac{-11v}{5} \, du\,dv}_{\substack{D \text{ is symm over } u\text{-axis} \\ \frac{-11v}{5} \text{ has odd symm.}}} = 0 + 0 = 0$$

2. *(20 pts)* The curve P is parametrized by $\vec{x}(t) = (t^2 + e^{t(t-1)}, t^4 - 3t^3 + 2t, \cos(\pi t))$, with $0 \le t \le 1$. Compute $\int_P \vec{F} \cdot \vec{T}\, ds$, where $\vec{F}(x,y,z) = \langle 3x^2y + ze^{xz}, x^3, xe^{xz} \rangle$.

$$\nabla \times \vec{F} = \left(0-0,\ (e^{xz} + xze^{xz}) - (e^{xz} + xze^{xz}),\ 3x^2 - 3x^2 \right)$$

$$= \vec{0} \implies \vec{F} \text{ is conservative}$$

$$f = \int 3x^2y + ze^{xz}\,dx = x^3y + e^{xz} + C_1(y,z)$$

$$f = \int x^3\,dy = x^3y + C_2(x,z)$$

$$f = \int xe^{xz}\,dz = e^{xz} + C_3(x,y)$$

We can use $f = x^3y + e^{xz}$.

$$\int_P \vec{F} \cdot d\vec{x} = f(\vec{x}(1)) - f(\vec{x}(0))$$

$$= f(2,0,-1) - f(1,0,1)$$

$$= (0 + e^{-2}) - (0 + e^{1})$$

$$= \boxed{\frac{1}{e^2} - e}$$

3. *(20 pts)* A gas cloud in space, with density given by $\delta = \delta(x, y, z)$, creates a gravitational field near the origin described by

$$\vec{F}_g = \left\langle \begin{matrix} 3 + 4x - 2y + 3x^2 - 4y^2 \\ 2 - 2x + 2y - z + xy - y^2 \\ 5 - x - y - 5z + xz - y^2 - z^2 \end{matrix} \right\rangle$$

Compute $\dfrac{\delta(1,0,0)}{\delta(0,0,0)}$.

$$\nabla \cdot \vec{F}_g = (4 + 6x) + (2 + x - 2y) + (-5 + x - 2z)$$

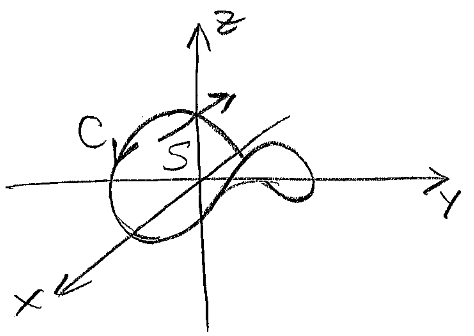
$$= 1 + 8x - 2y - 2z$$

$$\nabla \cdot \vec{F}_g = -4\pi G \delta, \quad \text{so} \quad \delta = \frac{\nabla \cdot F}{-4\pi G}$$

Then $\dfrac{\delta(1,0,0)}{\delta(0,0,0)} = \dfrac{\nabla \cdot F_g(1,0,0)}{\nabla \cdot F_g(0,0,0)} = \dfrac{9}{1} = 9$

4. *(20 pts)* The curve C is parametrized by $\vec{x}(t) = (\cos t, \sin t, \cos t \sin t)$, with $0 \le t \le 2\pi$. Compute the line integral along this curve of the vector field $\vec{F}(x,y,z) = \langle x^2 - 2y, 2e^y - z, z^3 + 3x \rangle$. *(Hint: You might be able to make use of an algebraic relationship between the functions describing x, y, and z on the curve.)*

On $\vec{x}(t)$, $z = xy$, so curve C is on this surface. It is closed, so $C = \partial S$, where S is oriented upward.

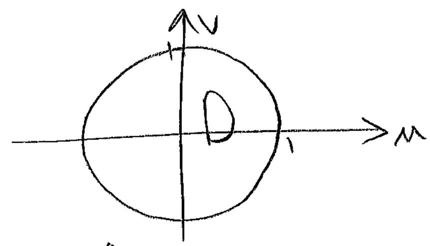


$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} u \\ v \\ uv \end{pmatrix}$$

$$\vec{N} = \vec{x}_u \times \vec{x}_v = \begin{pmatrix} 1 \\ 0 \\ v \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ u \end{pmatrix} = \begin{pmatrix} -v \\ -u \\ 1 \end{pmatrix}$$

(parametrization is oriented correctly!)

$$\int_C \vec{F} \cdot d\vec{x} = \iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \iint_D \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -v \\ -u \\ 1 \end{pmatrix} du\,dv$$



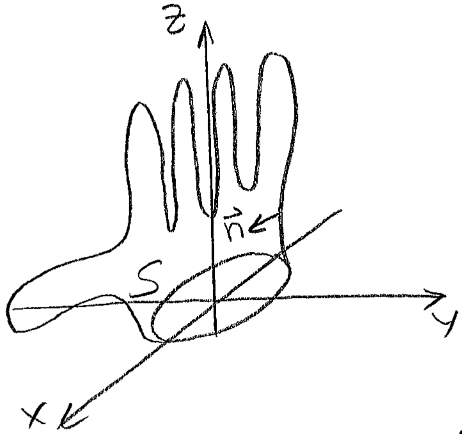
$$= \iint_D -v + 3u + 2 \, du\,dv$$

$$= \underbrace{\iint_D -v \, du\,dv}_{\substack{=0 \\ \text{by symm.}}} + \underbrace{\iint_D 3u \, du\,dv}_{\substack{=0 \\ \text{by symm.}}} + \underbrace{\iint_D 2 \, du\,dv}_{\substack{= 2(\text{area of } D) \\ = 2\pi}}$$

$$= 2\pi$$

5. *(20 pts)* The inward oriented surface S is in the shape of a left hand glove with the fingers pointing up, the thumb pointing in the direction of the x-axis, and the opening at the wrist is the curve in the xy-plane with equation $4x^2 + 9y^2 = 36$. Compute the flux through this surface of the vector field

$$\vec{F}(x, y, z) = \langle y^2 z - z^2, 4 - xy, 3 + xz \rangle$$



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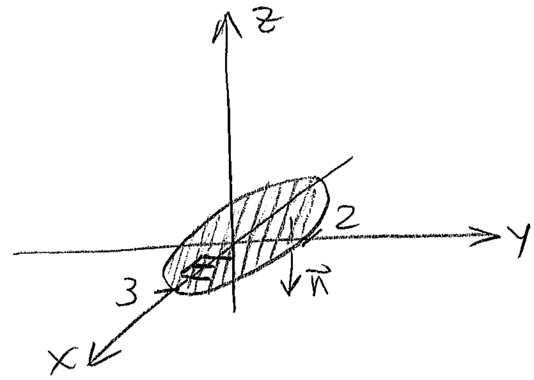

We can use surface independence because

$$\nabla \cdot \vec{F} = (0) + (-x) + (x) = 0.$$

Then

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_E \vec{F} \cdot d\vec{S} = \iint_E \vec{F} \cdot \vec{n}\, dS$$

$$= \iint_E \begin{pmatrix} y^2 z - z^2 \\ 4 - xy \\ 3 + xz \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} dS$$

$$= -\iint_E 3 + xz \, dS \qquad z = 0 \text{ on } E!$$

$$= -3 \cdot (\text{area of } E)$$

$$= -3\,(\pi \cdot 3 \cdot 2) = -18\pi$$