EXAM 3
Math 212, 2014-2015 Fall, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING. All answers must be simplified. All of the policies and guidelines on the class webpages are in effect on this exam.

Good luck!

Name ________________ Solutions

“I have adhered to the Duke Community Standard in completing this examination.”

Signature: ______________________

1. ____________

2. ____________

3. ____________

4. ____________

5. ____________

6. ____________

Total Score ____________ (/100 points)
1. (20 pts) The curve $C$ is parametrized by $\vec{x}(t) = (2t^2 - 4t, t + t^3, t^2)$, $0 \leq t \leq 1$, and the vector field $\vec{F}$ is given by $\vec{F}(\vec{x}) = (yze^{xy}, xze^{xy}, e^{xy})$. Compute $\int_C \vec{F} \cdot d\vec{s}$.

\[ \nabla \times \vec{F} = \vec{0}, \quad \text{so} \quad \vec{F} = \nabla f \quad \text{and we find $f$ by} \]

\[ \begin{align*}
  f &= \int yze^{xy} \, dx = ze^{xy} + c_1(y, z) \\
  f &= \int xze^{xy} \, dy = ze^{xy} + c_2(x, z) \\
  f &= \int e^{xy} \, dz = ze^{xy} + c_3(x, y) \\
\end{align*} \]

\[ \Rightarrow f = ze^{xy} \]

Then

\[ \int_C \vec{F} \cdot d\vec{s} = f(\vec{x}(1)) - f(\vec{x}(0)) \]

\[ = f(-2, 3, 1) - f(0, 0, 0) \]

\[ = e^{-4} - 0 = e^{-4} \]
2. (20 pts) The curve $P$ is the intersection of the paraboloid $z = x^2 + y^2$ and the plane $z = 2x + 4y$, oriented clockwise as seen from above. The vector field $\vec{G}$ is given by $\vec{G}(x) = (e^{x^2} + 3y + z, \sin(y^3) + x, 4y - 3z)$. Compute $\int_P \vec{G} \cdot \vec{T} \, ds$.

$\mathbf{x}^2 + y^2 = 2x + 4y$

$(x-1)^2 + (y-2)^2 = 5$

$P$ has a circular projection $D$ to $xy$-plane, and bounds a solid ellipse $E$ in the plane $z = 2x + 4y$ (oriented downward). Stokes's theorem then gives us

$$\int_P \vec{G} \cdot \vec{T} \, ds = \iint_E (\nabla \times \vec{G}) \cdot \vec{n} \, dS = \iint (4, 1, -2) \cdot \vec{n} \, dudw$$

$E$ is param. by $\mathbf{x} = (u, v, 2u + 4v)$, so

$$\mathbf{x}_u = (1, 0, 2) \quad \Rightarrow \quad \vec{n} = (-2, -4, 1)$$

$$\mathbf{x}_v = (0, 1, 4)$$

Then the integral becomes

$$\iint (4, 1, -2) \cdot (-2, -4, 1) \, dudw = -\iint (-14) \, dudw$$

$\{bc of opposite orientation\}$

$= 14 \text{ (area of } D)$

$= \frac{70}{\pi}$
3. (20 pts) The torus \( T \) has spherical equation \( \rho = \sin \phi \) and is oriented inward, and the vector field \( \vec{H} \) is given by \( \vec{H}(\vec{x}) = (2x - e^y, -y + 2yz, z - z^2) \). Compute \( \iint_T \vec{H} \cdot \vec{n} \, dS \).

\( T \) is the boundary (with opposite orientation) of the solid donut \( D \). Then by Gauss's theorem,

\[
\iiint_D \vec{H} \cdot \vec{n} \, dV = -\iiint_D \nabla \cdot \vec{H} \, dV
\]

\[= -2 \text{ (volume of } D)\]

\( D \) is rotation of disk of radius \( \frac{1}{2} \) (area = \( \frac{\pi}{4} \)) around \( z \)-axis (dist. = \( \pi \))

So by Pappus's theorem, \( V = \frac{\pi^2}{4} \)

Then \( \Phi = -2V = -\frac{\pi^2}{2} \)
4. (10 pts) Perfectly balanced balloons are being pushed and spun through the air by the air flow described by the vector field \( \vec{A}(\vec{x}) = (z - xz, 2x + yz, xy) \).

(a) For a balloon at the point (1, 4, 1), find the velocity, and the axis of rotation of its spin.

\[
\vec{J} = \vec{A}(1, 4, 1) = \begin{bmatrix} 0, 6, 4 \end{bmatrix}
\]

\( \nabla \times \vec{A} = (-x-y, 1-x-y, 2) \)

\( \nabla \times \vec{A}(1, 4, 1) = (-3, -4, 2) \)

axis is in direction of this vector.

(b) What is the ratio of the spinning rates of balloons at (1, 4, 1) and (0, 0, 0)?

\[
\|
\nabla \times \vec{A}(1, 4, 1)\n\| = \|
(-3, -4, 2)\n\| = \sqrt{29}
\]

\[
\|
\nabla \times \vec{A}(0, 0, 0)\n\| = \|
(0, 1, 2)\n\| = \sqrt{5}
\]

ratio = \( \sqrt{\frac{29}{5}} \)

5. (10 pts) Find the steady state current flow field \( \vec{J} \) that generates the magnetic field \( \vec{B} = (0, -ze^{-y^2-z^2}, ye^{-y^2-z^2}) \).

Steady state current flows \( \vec{J} \) generate magnetic fields \( \vec{B} \) that relate by \( \nabla \times \vec{B} = \mu_0 \vec{J} \)

\[
S_0 \vec{J} = \frac{1}{\mu_0} \nabla \times \vec{B}
= \frac{1}{\mu_0} \begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
\end{pmatrix}
\begin{pmatrix}
e^{-y^2-z^2} + y(-2ye^{-y^2-z^2}) - (-1e^{-y^2-z^2} - z(-2ze^{y^2-z^2})) \\
\text{0} \\
\text{0} \\
\end{pmatrix}
\]

\[
= \frac{1}{\mu_0} \begin{pmatrix}
(2-2y^2-2z^2)e^{-y^2-z^2} \\
\text{0} \\
\text{0} \\
\end{pmatrix}
\]
6. (20 pts) The surface $H$ is the part of the hyperboloid of one sheet with equation $x^2 + y^2 - z^2 = 9$ that is between $z = -4$ and $z = 4$, oriented toward the $z$-axis. The vector field $\vec{F}$ is given by $\vec{F}(\vec{x}) = (x(x^2 + y^2), y(x^2 + y^2), -4z(x^2 + y^2))$. Compute $\iint_H \vec{F} \cdot \vec{n} \, dS$.

\[
\nabla \cdot \vec{F} = (3x^2 + 4y) + (x^2 + 3y^2) + (-4(x^2 + y)^2)
\]
\[= 0\]

So $\vec{F}$ is surface independent.

$H$ has the same boundary as the cylinder $C$, so we can compute as

\[
\iiint_C \vec{F} \cdot \vec{n} \, dS
\]

On $C$, $x^2 + y^2 = 25$, so

\[
\vec{F} = \langle 25x, 25y, -100z \rangle
\]

Also, using symmetries of the cylinder, $\vec{n} = \langle -\frac{x}{5}, -\frac{y}{5}, 0 \rangle$

Then

\[
\mathcal{F} = \iiint_C \left( \begin{array}{c} 25x \\ 25y \\ -100z \end{array} \right) \cdot \left( \begin{array}{c} -\frac{x}{5} \\ -\frac{y}{5} \\ 0 \end{array} \right) \, dS = \iiint_C -5(x^2 + y^2) \, dS
\]

\[
= \iint (-125) \, dS = -125 \, (\text{area of } C)
\]

\[
= -125 \left( 2\pi \cdot 5 \cdot 8 \right) = -10,000 \pi
\]