EXAM 3

Math 212, 2013-2014 Spring, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.
All answers must be simplified. All of the policies and guidelines
on the class webpages are in effect on this exam.

Good luck!

Name Solutions

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ______________________

1. ____________

2. ____________

3. ____________

4. ____________

5. ____________

6. ____________

Total Score ____________ (/100 points)

1. *(15 pts)* Compute $\int_C \vec{F} \cdot \vec{T} ds$, where $\vec{F}(x,y) = (2x\sin(y^2), 2x^2y\cos(y^2))$, and the curve C is described by

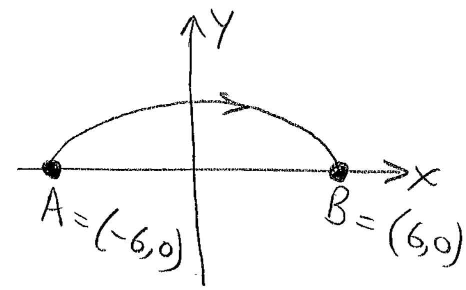
$$\frac{x^2}{36} + \frac{y^2}{4} = 1 \quad \text{and} \quad y \geq 0, \text{ oriented to the right}$$

$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 4xy\cos(y^2) - 4xy\cos(y^2) = 0$; we note

and we note $\vec{F} = \nabla f$ with $f = x^2\sin(y^2)$.

Then $\int_C \vec{F} \cdot d\vec{x} = f(B) - f(A)$

$= 0 - 0 = 0$



2. *(15 pts)* Compute $\int_C \vec{F} \cdot \vec{T} ds$, where $\vec{F}(x,y) = (y\cos(xy) + y, x\cos(xy) - x)$, and the curve C is the unit square oriented clockwise.

$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = (\cos xy - xy\sin xy - 1) - (\cos xy - yx\sin xy + 1)$

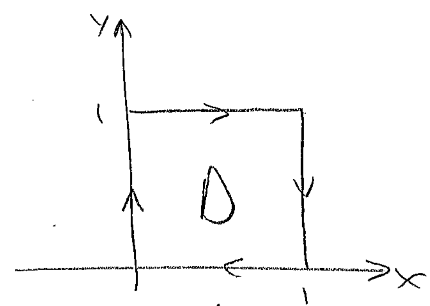
$= -2$

$\int_C \vec{F} \cdot d\vec{x} = -\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dx\,dy$

$= 2\iint_D dx\,dy$

$= 2$ (area of D)

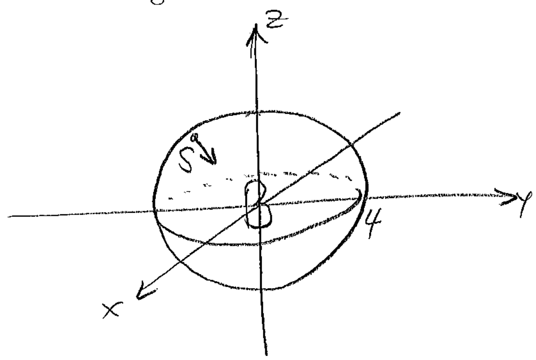
$= 2$



because of clockwise or. of C.

3. *(15 pts)* Compute the flux of the vector field $\vec{F}(x,y,z) = (x + xy, x + xz, y + yz)$ through the inward oriented sphere of radius 4 centered at the origin.

$$S = -\partial B$$



$$\nabla \cdot \vec{F} = (1+y) + (0) + (y)$$

$$= 1 + 2y$$

$$\iint_S \vec{F} \cdot \vec{n}\, dS = -\iiint_B \nabla \cdot \vec{F}\, dV$$

$$= -\iiint_B (1+2y)\, dV$$

$$= -\iiint_B 1\, dV \underbrace{- \iiint_B 2y\, dV}_{= 0 \text{ by symmetry through } xz\text{-plane}}$$

$$= -(\text{vol. of } B)$$

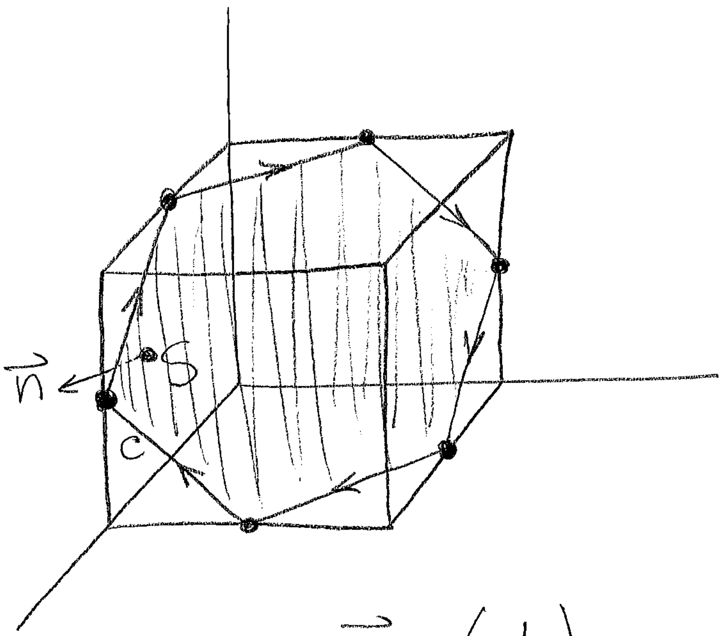
$$= -\frac{4}{3}\pi (4)^3$$

$$= \frac{-256\pi}{3}$$

4. *(20 pts)* The curve C is the regular hexagon in the plane $x+y+z=3$ with vertices at $(2,0,1)$, $(1,0,2)$, $(0,1,2)$, $(0,2,1)$, $(1,2,0)$, $(2,1,0)$, oriented in the direction that these vertices are listed. Compute the circulation around this curve of the vector field $\vec{F}(x,y,z) = (xy^2z^2+y, x^2yz^2+z, x^2y^2z+x)$.

$C = \partial S$

$$\begin{pmatrix}-1\\-1\\-1\end{pmatrix}/\sqrt{3} = \vec{n}$$

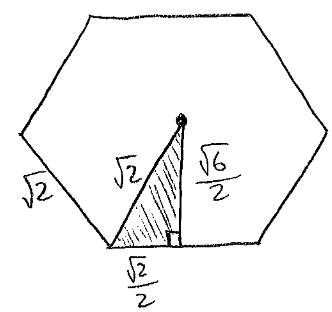


$$\int_C \vec{F}\cdot d\vec{x} = \iint_S (\nabla\times\vec{F})\cdot\vec{n}\, dS \qquad \nabla\times\vec{F} = \begin{pmatrix}-1\\-1\\-1\end{pmatrix}$$

$$= \iint_S \begin{pmatrix}-1\\-1\\-1\end{pmatrix}\cdot\begin{pmatrix}-1\\-1\\-1\end{pmatrix}/\sqrt{3}\, dS$$

$$= \sqrt{3}\ (\text{area of hexagon})$$

$$= \sqrt{3}\left(12\left(\frac{1}{2}\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{6}}{2}\right)\right)\right)$$



$$= 9$$

5. *(15 pts)* Compute the flux of the vector field $\vec{F}(x,y,z) = (2,2,3)$ through the surface that is parametrized by $\vec{x}(u,v) = (u+v, u-v, uv)$, $0 \le u \le 1$, $0 \le v \le 1$.

$$\vec{X}_u = \begin{pmatrix} 1 \\ 1 \\ v \end{pmatrix} \quad \vec{X}_v = \begin{pmatrix} 1 \\ -1 \\ u \end{pmatrix} \quad \vec{N} = \begin{pmatrix} u+v \\ v-u \\ -2 \end{pmatrix}$$

$$\Phi = \iint \vec{F} \cdot \vec{N} \, du\,dv = \iint \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} u+v \\ v-u \\ -2 \end{pmatrix} du\,dv$$

$$= \int_0^1 \int_0^1 4v - 6 \, du\,dv$$

$$= \int_0^1 \left(4uv - 6u\right]_{u=0}^{u=1} dv$$

$$= \int_0^1 4v - 6 \, dv$$

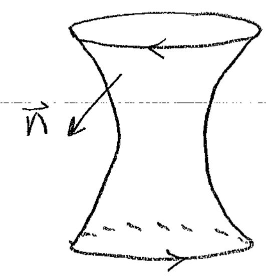
$$= \left(2v^2 - 6v\right]_0^1$$

$$= -4$$

6. *(20 pts)* The surface S is described in cylindrical coordinates by $r = 1 + z^2$, $-1 \leq z \leq 1$, and is oriented outward. Compute the flux through S of the vector field

$$\vec{F}(x, y, z) = ((x^2 + y^2 - 4)x, (x^2 + y^2 - 4)y, -4z(x^2 + y^2 - 2))$$

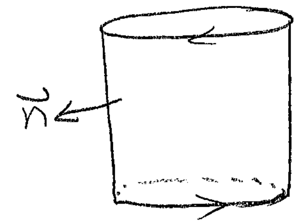
$$\vec{F} = \begin{pmatrix} x(x^2+y^2-4) \\ y(x^2+y^2-4) \\ -4z(x^2+y^2-2) \end{pmatrix}$$



$$\nabla \cdot \vec{F} = (3x^2+y^2-4) + (x^2+3y^2-4) + (-4(x^2+y^2-2)) = 0$$

So $\vec{F}$ is surface independent. Consider instead

$S_1 = \{ r = 2 \ , \ -1 \leq z \leq 1 \}$



with same boundary as S

Unit normal is $\vec{n} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} / 2$, and on S_2 we have $x^2+y^2 = r^2 = 4$, so $\vec{F} = (0, 0, -8z)$

Then

$$\Phi_S = \Phi_{S_1} = \iint_{S_1} \begin{pmatrix} 0 \\ 0 \\ -8z \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} / 2 \, dS$$

$$= \iint_{S_1} 0 \, dS = 0.$$