EXAM 2

Math 212, 2013-2014 Spring, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.
All answers must be simplified. All of the policies and guidelines
on the class webpages are in effect on this exam.

Good luck!

Name Solutions

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ______________________

1. ____________

2. ____________

3. ____________

4. ____________

5. ____________

6. ____________

7. ____________

Total Score ____________ (/100 points)

1. (10pts) Find the equation of the tangent plane to the surface $\underbrace{x^2 + y^4 - z^2}_{f(x,y,z)} = 8$ at the point $\underbrace{(1, 2, 3)}_{\vec{x}_0}$.

$$\nabla f = \begin{pmatrix} 2x \\ 4y^3 \\ -2z \end{pmatrix} \qquad \nabla f\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 32 \\ -6 \end{pmatrix}$$

this is $\perp$ to tangent plane, so we can choose any multiple of this as $\vec{n}$.

We choose $\vec{n} = (1, 16, -3)$, and $\vec{x}_0 = (1,2,3)$. Then

$\vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{x}_0$ becomes $\boxed{x + 16y - 3z = 24}$

2. (10pts) Suppose that z is a differentiable function of x and y, and that x and y are differentiable functions of s and t. For certain values of s and t, we know that $\frac{\partial z}{\partial x} = 3$, $\frac{\partial z}{\partial y} = 2$, $\frac{\partial z}{\partial s} = 5$, and $\frac{\partial y}{\partial s} = 4$. Compute $\frac{\partial x}{\partial s}$.

$$\begin{tikzcd}
 & z & \\
x \arrow[ur] & & y \arrow[ul] \\
s \arrow[u] \arrow[urr] & & t
\end{tikzcd}$$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

Then at the given point,

$$5 = 3\frac{\partial x}{\partial s} + (2)(4)$$

$$\Rightarrow \boxed{\frac{\partial x}{\partial s} = -1}$$

3. *(10pts)* Suppose $\underbrace{3xy + y^2z - 2z^2 + 9}_{F(x,y,z)} = 0$. At the point $(2,1,3)$, what is $\frac{\partial y}{\partial z}$? *(Be sure to explain all necessary steps!)*

$$\frac{\partial F}{\partial y} = 3x + 2yz \qquad \frac{\partial F}{\partial y}(2,1,3) = 12 \neq 0$$

$\Rightarrow$ y is a function of x, z near $(2,1,3)$.

$$\frac{\partial y}{\partial z} = -\frac{\partial F/\partial z}{\partial F/\partial y} = -\frac{y^2 - 4z}{3x + 2yz}, \quad \boxed{\frac{\partial y}{\partial z}(2,1,3) = \frac{11}{12}}$$

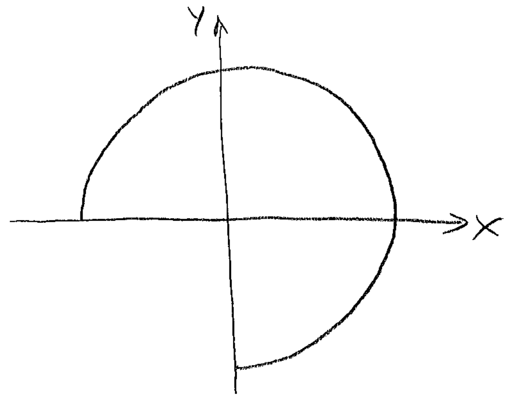
4. *(10pts)* The function $f : \mathbb{R}^2 \to \mathbb{R}$ is given by $f(x,y) = x^3y^2 - 3xy$. The point $\vec{x} = (x,y)$ is at $(1,2)$, and moving with speed 3 in the direction of fastest increase of f. Compute $\frac{df}{dt}$.

$$\nabla f = \begin{pmatrix} 3x^2y^2 - 3y \\ 2x^3y - 3x \end{pmatrix} \qquad \nabla f(1,2) = \begin{pmatrix} 6 \\ 1 \end{pmatrix}$$

Then $\frac{df}{ds}$ in this direction is $\|\nabla f\| = \sqrt{37}$

And $\frac{df}{dt} = \frac{df}{ds}\underbrace{\frac{ds}{dt}}_{\text{speed}=3} = (\sqrt{37})(3) = 3\sqrt{37}$

5. *(15pts)* A population of bacteria is spread over the part of the unit disk where either x <u>or</u> y (or both) is greater than or equal to zero. The population density (in millions per unit area) is given by $\delta(x, y) = x^2$. Compute the total population of bacteria.



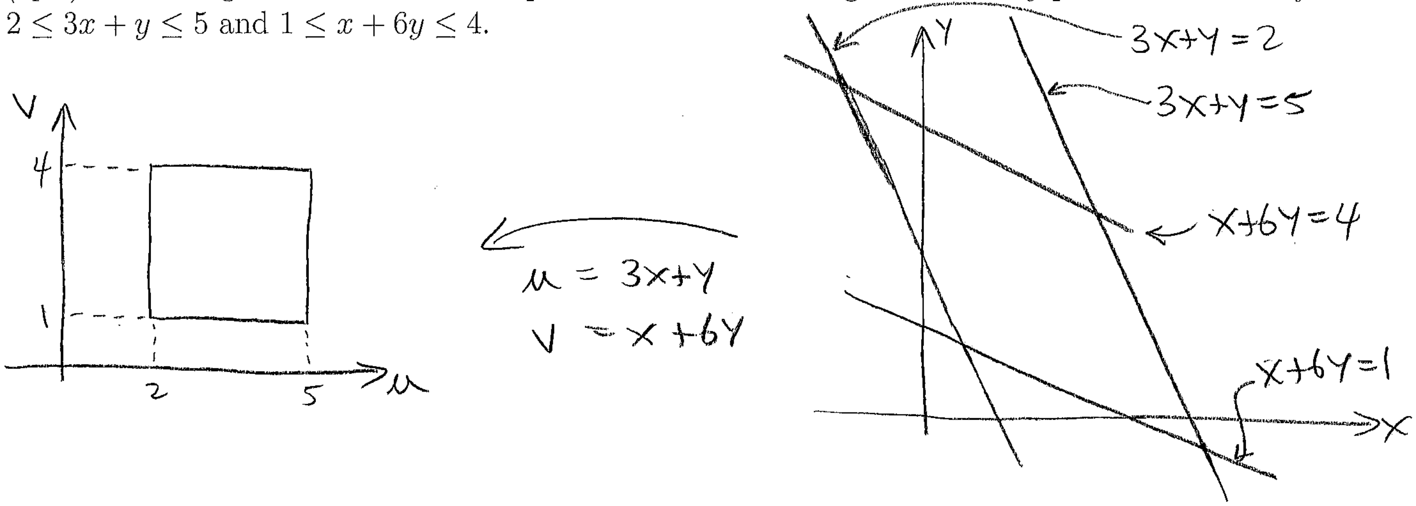
$$P = \iint_D \delta \, dA = \iint_D x^2 \, dx\,dy$$

$$= \int_{-\pi/2}^{\pi} \int_0^1 (r\cos\theta)^2 \, r\,dr\,d\theta$$

$$= \int_{-\pi/2}^{\pi} \left(\frac{1}{4} r^4 \cos^2\theta\right)\Big]_0^1 \, d\theta = \int_{-\pi/2}^{\pi} \frac{1}{4}\cos^2\theta \, d\theta$$

$$= \int_{-\pi/2}^{\pi} \frac{1 + \cos 2\theta}{8} \, d\theta = \frac{\theta}{8} + \frac{\sin(2\theta)}{16}\Big]_{-\pi/2}^{\pi} = \frac{3\pi}{16}$$

6. *(15pts)* Use change of variables to compute the area of the region in the xy-plane described by $2 \leq 3x + y \leq 5$ and $1 \leq x + 6y \leq 4$.



$$\int_1^4 \int_2^5 (1) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv = \iint_R 1 \, dx\,dy = A$$

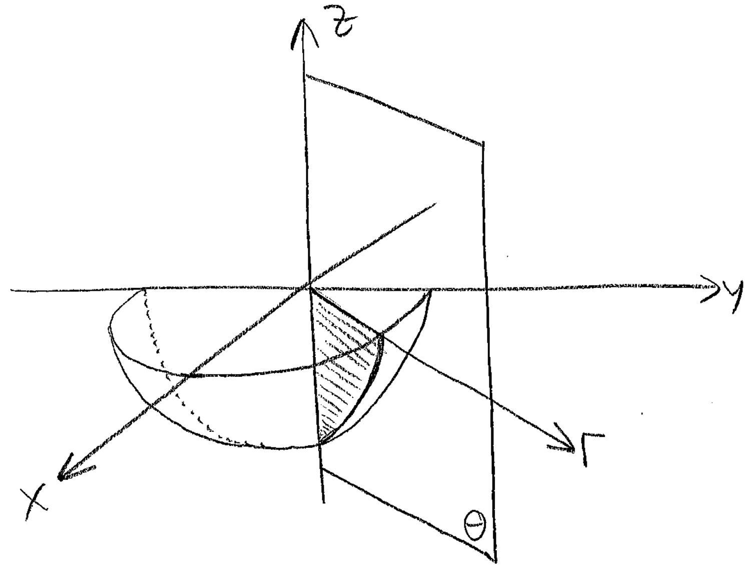
$$\left(\frac{\partial(u,v)}{\partial(x,y)} = \det\begin{pmatrix} 3 & 1 \\ 1 & 6 \end{pmatrix} = 17, \text{ so } \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{17}\right)$$

$$= \int_1^4 \int_2^5 \frac{1}{17} \, du\,dv = \frac{9}{17}$$

7. (5pts) The domain D consists of those points with $x^2 + y^2 + z^2 \leq 9$, $x \geq 0$, $z \leq 0$. Set up, but do not evaluate, a triple integral in spherical coordinates representing $\iiint_D f\, dV$.

Per the figure,

$\theta_1 = \frac{-\pi}{2}$, $\theta_2 = \pi/2$



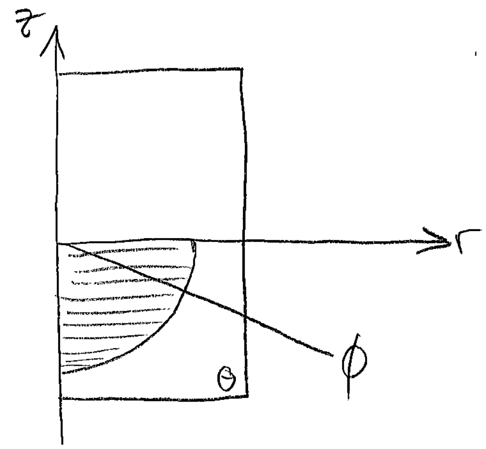
In this θ cross-section,

we see

$\phi_1 = \frac{\pi}{2}$, $\phi_2 = \pi$

and

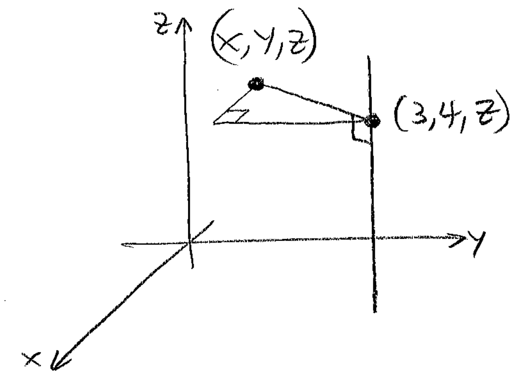
$\rho_1 = 0$, $\rho_2 = 3$



So

$$\iiint_D f\, dV = \int_{-\pi/2}^{\pi/2} \int_{\pi/2}^{\pi} \int_0^3 (f)\, \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

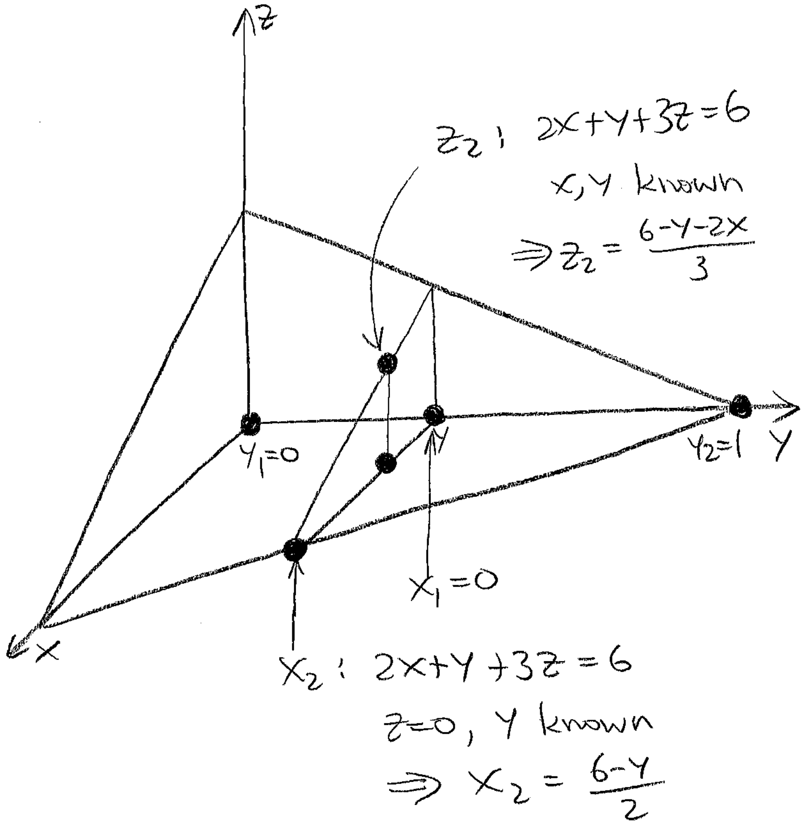
8. *(5pts)* The line L is parallel to the z-axis and passes through $(3,4,0)$. The solid tetrahedron T is bounded by the coordinate planes and $2x+y+3z=6$, with density given by $\delta(x,y,z)=1+x^2$. Set up, but do not evaluate, a triple integral in rectangular coordinates that represents the moment of inertia of T around L.



$$r^2 = (x-3)^2 + (y-4)^2$$

$$I = \iiint_T r^2 \, dm$$

$$= \iiint_T r^2 \delta \, dV$$



Doing y-slices first:

$y_1 = 0$, $y_2 = 1$

Then x-slices:

$x_1 = 0$, $x_2 = \frac{6-y}{2}$

Then z-slices:

$z_1 = 0$, $z_2 = \frac{6-y-2x}{3}$

So

$$I = \iiint_T r^2 \delta \, dV = \int_0^6 \int_0^{\frac{6-y}{2}} \int_0^{\frac{6-y-2x}{3}} \left((x-3)^2 + (y-4)^2\right)(1+x^2) \, dz \, dx \, dy$$